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K. Siegbahn and K. Edvarson, Beta-ray spectroscopy in the precision range of 1:10^5 137

E. Sokolowski, K. Edvarson and K. Siegbahn, Relative conversion probabilities in the K, L_I and L_{II} shells of the 238,6 keV-line of ThB 160

K. McVoy and H. Steinwedel, Principal axis transformation for a free nucleon coupled to an uncharged scalar meson field 164

J. E. Moyal, Theory of the ionization cascade 180

P. S. Goel, S. Jha, D. Lal, P. Radhakrishna and Rama, Cosmic ray produced beryllium isotopes in rain water. 196

J. Mc Connell, Theory of the negative proton. 202

Book reviews. 229

News items 231

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β-RAY SPECTROSCOPY IN THE PRECISION RANGE OF 1 : 10^5

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Abstract: An iron-free double focusing β spectrometer for precision work is described. Several conversion lines in the spectrum of Th B + C have been measured absolutely, using a method which eliminates the measurement of the magnetic field. By combining several such β spectroscopic data one can caculate the elementary constant combination h/m_0c. The result is in excellent agreement with results obtained by other means. This is a sensitive check on the accuracy of the individual $B\varrho$ values of the different lines.

1. Introduction

The accuracy in determination of β-line energies has hitherto usually not exceeded 1 : 10^3 except for a few cases, when an accuracy of a few parts in 10^4 has been reported. Even when a resolving power of the order of 1 : 10^3 or somewhat better has been achieved it has not been possible to fully utilize this feature in the energy determination because of the difficulties in determining the line positions and the magnetic field. Furthermore, in spectrometers like the semicircular, a high resolution can only be obtained at a great sacrifice in transmission, which severely limits the practical possibilities for work of this character. For several types of problems it is desirable to be able to make β-energy determinations with an accuracy of some parts in 10^5, i.e. about the same as in X-ray spectroscopy. This would be possible in an instrument with a resolving power of about 1 : 10^4 and an accuracy in $B\varrho$-determination of about 1 : 10^5. Such an instrument would also be of value for intensity determinations of closely situated lines, e.g. L-conversion lines.

The double-focusing spectrometer[1, 2]) offers the advantage of high resolving power at fairly high transmission. In the instruments of this type built hitherto, the necessary inhomogeneous magnetic field has been achieved by means of magnets with suitably shaped pole-faces. This method of creating the field has certain disadvantages,

due to remanence and permeability-change effects, which make high-precision measurements very difficult, especially at low β-energies. Thus the field strength is not an unequivocal function of the exciting current and further the field form is to some extent dependent on the magnetization state. The first of these effects necessitates a rather cumbersome precision determination of the field strength at each measurement. The second will affect the image profiles and thus the peak values and intensities.

The above mentioned drawbacks are avoided if no ferromagnetic material is used in the construction of the spectrometer. In the present paper an *ironfree double-focusing spectrometer* is described. In this instrument, the desired magnetic field is created in the space between two coaxial solenoids and ferromagnetic material has been wholly avoided.

2. Theory of the Coil System

The theory of the electron-optical properties of axially-symmetric magnetic fields has been reviewed by several authors† and we will in this connection only discuss some points of interest for the present construction.

The desired field has a symmetry plane perpendicular to the symmetry axis and the field strength in this plane ($z = 0$) is

$$B = B_0\left[1 + \alpha\left(\frac{r - r_0}{r_0}\right) + \beta\left(\frac{r - r_0}{r_0}\right)^2 + \ldots\right]. \qquad (1)$$

The condition for double-focusing is $\alpha = -1/2$, whereas the β-value affects the aberration properties of the system. It is found[4]) that an electron of momentum $B_0 r_0$ departing from a point $(r_0 + \delta r,\ \delta z)$ in the source plane ($\phi = 0$) will strike the image plane ($\phi = \pi\sqrt{2}$) in $(r^*,\ z^*)$ given by

$$r^* = r_0 - \delta r + \frac{2 - 4\beta}{3r_0}\delta r^2 + \frac{4\beta - 3}{3r_0}\delta z^2 + \frac{2 - 16\beta}{3} r_0\gamma_r^2 + \frac{16\beta - 6}{3} r_0\gamma_z^2, \qquad (2)$$

$$z^* = -\delta z + \frac{8\beta}{3}\frac{\delta r \delta z}{r_0} + \frac{32\beta - 12}{3} r_0\gamma_r\gamma_z, \qquad (3)$$

where γ_r and γ_z are the radial and axial angles determining the initial direction of the trajectory relative to that of the median trajectory ($r = r_0,\ z = 0$).

† For a complete list of references cf. Beta- and Gamma-ray Spectroscopy, ed. K. Siegbahn, North Holland Publishing Company, Amsterdam 1955, Ch. III.

In order to get a high resolution at a fairly high transmission, it is necessary to choose a β-value giving a low value of the γ^2-terms in (2) at a fairly large value of γ_r, γ_z. As is pointed out in ref. [2]), this is possible at the β-values 1/8 and 3/8. In the first case, the γ_r^2-term will vanish, in the second, the γ_z^2-term.

In the choice between these values it is found that 3/8 generally is the more advantageous. Thus there is an astigmatism of the first order of the magnitude $(3-8\beta)\Delta r/2$ if source and detector are situated on the circle $r = r_0 + \Delta r$ instead of $r = r_0$ and this evidently will vanish at $\beta = 3/8$. Further it is necessary to have a not too small height of the source in order to get a reasonable luminosity, as the width must necessarily be small. If the ends of a source are to be used, however, the axial aperture must be at least about $h/2r_0\sqrt{2}$ (where h is the source height).

In the present case, there was a third reason for the choice of $\beta = 3/8$. The preliminary calculations showed that the desired field could be achieved between two coaxial solenoids. In that case space and constructional reasons favour an arrangement with a small variable radial aperture, whereas the axial aperture could be rather large.

As regards the dimensions, r_0 was not to be made too small, as the dispersion is linearly dependent on it. As the sources must be very thin at high precision work, it is necessary to be able to have a rather great source area. At given transmission and resolution, the source area is directly proportional to r_0. On the other hand, a large r_0-value means that B will be small at low β-energies, which means that small fluctuations in the external field may effect the measurements. A r_0-value of 30 cm was regarded as a reasonable compromise. This value will give a dispersion of about $1200\,\Delta B/B$ mm. Thus a movement of an image by 0.1 mm corresponds to $\Delta B/B = 0.08\,{}^0/_{00}$. Furthermore a β-energy of 4 keV, corresponding to $B\varrho = 214$ gauss cm, gives $B = 7$ gauss. The fluctuations in the external field will hardly exceed $\pm\, 10^{-4}$ gauss, giving an uncertainty in B of about 1 : 10^5, a permissible value. The greatest attainable $B\varrho$-value ought to be at least 2000 gauss cm.

From these considerations it was decided to construct an arrangement of ironfree coils giving a field with $\alpha = -1/2$ and $\beta = 3/8$ permitting a maximum B_0-value of at least 70 gauss. The result of the calculations indicated that the desired field could be created between two coaxial, countercoupled solenoids

with constant, but unequal current densities, provided certain conditions regarding the parameters r_0, r_1 and r_2 (radii of inner and outer coils respectively), h_1 and h_2 (half heights of the coils) and λ (Ampere-turns ratio inner to outer) were satisfied. Of these parameters r_0 was fixed to 30 cm and the values 24 and 36 cm were chosen for r_1 and r_2 respectively. As it was very difficult to obtain the condition in an explicit form, numerical solutions were worked out for several values of h_1, h_2 and λ and an approximate solution with $h_1 = h_2 = 24.3$ cm and $\lambda = 5.00$ was obtained by graphical interpolation.† In this case 70 gauss is obtained at about 10^4 At in the inner coil. The solution with $h_1 = h_2 = h$ has the advantage that the second gradient is independent of λ. It is thus possible to make a final adjustment of the first gradient by changing λ without changing the second gradient. Furthermore this solution has evident constructional advantages.

3. Construction of the Coil System and Magnetic Field Measurements

The construction of the coil system is shown in figs 1 and 2. The coil frames are made of 10 mm copper sheet and machined

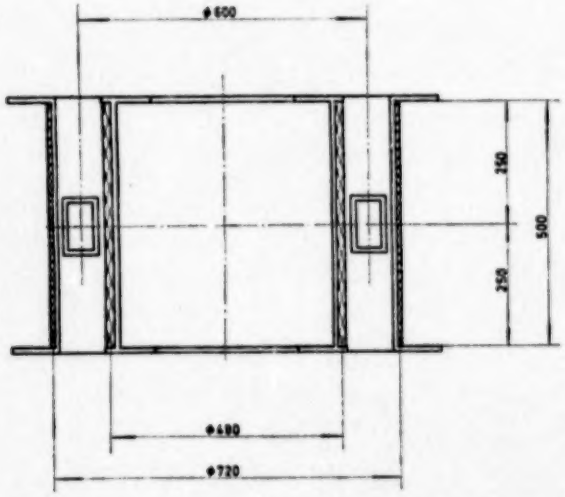

Fig. 1. The main features of the spectrometer system

to a radius accuracy of $\pm$ 0.1 mm. Coils for water cooling (not shown in the figures) are soldered to the inner walls of the frames.

† We want to thank Dr L. Dreyfus, ASEA, who kindly carried out these calculations.

The coils are wound with rectangular copper wire, cotton insulated, the inner with 357 turns of 2 wires (2.4 × 3.6 mm) in

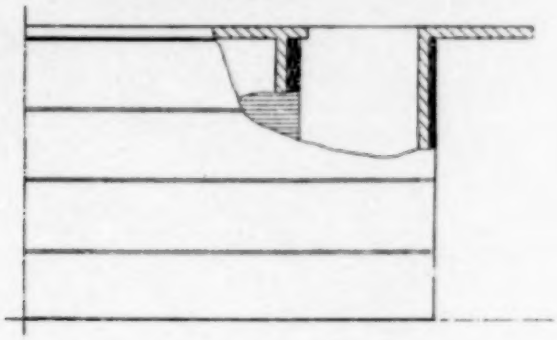

Fig. 2. Detail of the coil system, showing the main windings (alternate turns black and white) and the extra windings (cf. text) consisting of broad copper strips

parallel in 4 layers and the outer with 71 turns of 4 wires (3.2 × 1.7 mm) in parallel in 2 layers. The coils are fixed to a triangle made of channel section aluminium. The data of the coils are summarized in table 1.

TABLE 1

Data of the Spectrometer Coils

	Inner coil	Outer coil
Number of turns	357	71
Mean radius of windings	240 mm	360 mm
Height of windings	486 mm	488 mm
Copper dimensions	2.4 × 3.6 mm 4 layers, 2 wires in parallel	3.2 × 1.7 mm 2 layers, 4 wires in parallel
Resistance	0.523 Ω	0.127 Ω
Effect at $B_0 \approx 70$ gauss	275 W	67 W
Spectrometer constant	78 $B\varrho$ amp^{-1}	

The determination of the field form was made with a test coil at a field strength of about 20 gauss. Measurements were made in the interval $r = 27$ to 32.5 cm, at $z = 0$ and + 4 cm and corrections were made for the test coil dimensions.

The main purpose of the measurement was to determine the λ-value giving $\alpha = -1/2$, as the numerical calculations only had

given approximate values of λ and h. For that reason λ was varied by shunting one of the coils. In fig. 3 are shown the results of three determinations at different λ-values compared with the ideal field form. Evidently $\lambda = 4.52$ gives a very good agreement with

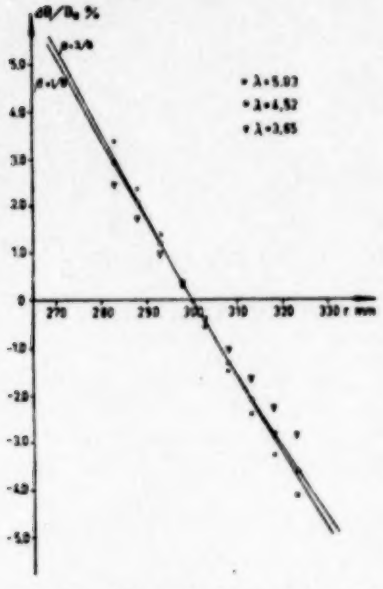

Fig. 3. Determination of the field form at different λ-values, compared with theoretical field form

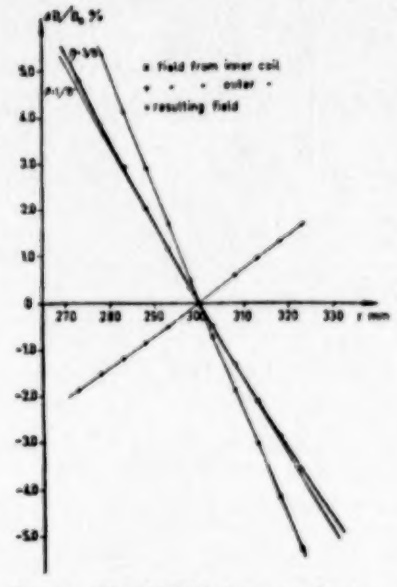

Fig. 4. Final field form and contributions from each coil

the ideal field. With the coils in series, the original system gave $\lambda = \frac{357}{71} = 5.03$. As it seemed undesirable to shunt the inner coil, it was decided to wind the outer with 8 extra turns of broad copper strip, giving $\lambda = 4.52$. In fig. 4 the final field form and the contributions from each coil are given.

DESIGN OF THE INSTALLATION

The design of the main parts of the spectrometer is shown in figs 1, 2, 5 and 7. The spectrometer is mounted on an aluminium framework, on the top of which a granite ring is fixed. The triangle supporting the coil system rests by means of three screws on three aluminium plates fixed to the granite ring. With these screws the axial adjustment of the spectrometer is performed. The vacuum-chamber is provided with three perpendicular aluminium bars with plates in their lower ends. The plates rest on the granite ring and the radial adjustment is made by screws in these plates.

The vacuum-chamber is made of 10 mm copper. It is ringshaped

with inner radius 270 mm, outer radius 330 mm and height 80 mm (inside dimensions). On the edges of the chamber are grooves for 0-rings. The lid of the chamber (fig. 6) is also made of 10 mm copper. From it are suspended the sample holder system, the shutters and the detector.

The sample holder system consists of a lock device (which can

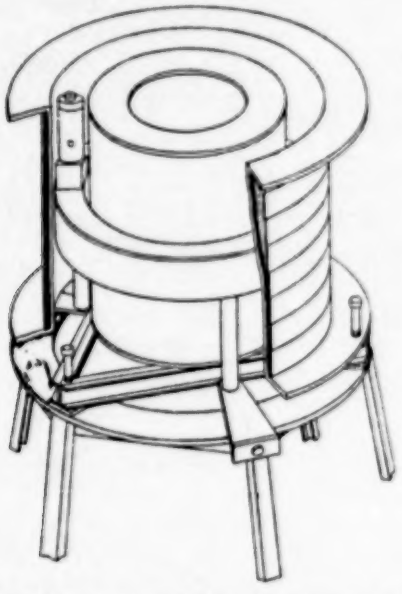

Fig. 5. Main parts of the spectrometer. Outer coil cut in order to show mounting of coils and vacuum chamber

be connected to the fore vacuum system), through which a copper tube ($\phi = 35$ mm) is introduced into the vacuum chamber. A frame supporting the sample is attached to the end of the tube. This frame is movable by about $\pm$ 10 mm in a direction at right angles to the spectrometer radius.

There are four shutters, of which the second is adjustable. The radial adjustment is made from the outside and the shutter position is read on a scale. The axial adjustment can only be made when the lid is removed. When this shutter is set for maximum transmission, the radial aperture is 0.062 radians and the axial 0.098, giving a solid angle of 0.2 %.

The detector is a GM-counter with side-window, provided with four slits ranging from 0.2 mm $\times$ 10 mm to 2 mm $\times$ 10 mm, interchangeable from the outside. The counter is connected to a filling device of conventional type.

The system is evacuated by an oil-diffusion pump made of aluminium (capacity 200 l sec^{-1}), placed in the stand below the vacuum chamber and connected to the latter by metallic bellows ($\phi = 50$ mm). The diffusion pump is connected to the fore vacuum pump by a 3 m copper tube ($\phi = 100$ mm). The earth's magnetic field is compensated by a system of three pairs of Helmholtz coils with a radius of 1.6 m, set in the vertical, N - S and E - W directions and with their centres coinciding with the centre of the spectrometer. (At first, only the vertical and N - S coils were used, but it was found that a third coil faciliated the compensation). To test the effectiveness of the compensation, a magnetometer Type 1.100 from Institut Dr Förster is used. With this instrument, which has a maximum sensitivity of 10^{-5} gauss, the resultant field at the centre of the spectrometer is found to be less than 10^{-4} gauss.

An investigation of the remaining axial field along the electron-path was made by measuring the difference in field strength between the centre and points along the path when the field is compensated at the centre. The result of two series of measurements is shown in fig. 8. The sinusoidal variation found, which was probably due

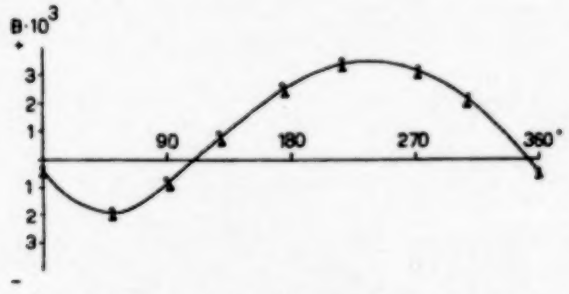

Fig. 8. Measurement of the remaining field along the electron path when the earth's magnetic field is compensated at centre of the spectrometer

to imperfections in the Helmholtz system, was sufficiently great to affect low energy measurements, if no precautions were taken. For that reason, the chamber lid was orientated in such a way that the residual field was in opposite directions in the two halves of the electron trajectory from source to detector. An investigation of the remaining horizontal field showed that this had no significant effect even at very low β-energies.

Some measurements of the remanence effects were also made. A current of about 25 A was run through the spectrometer, first in

one direction and then in the other, and the Helmholtz currents necessary for compensation of the earth's field were determined after each run. No significant difference was detected.

The current control is performed with an „AEG Messtisch mit grossem Präzisions-Gleichstrom-Kompensator" permitting a maximum current of 50 A. This instument permits an accuracy of about 1 part in 10^5 if the Weston cell and the standard resistances are held at constant temperature. For that reason they are placed in an oil thermostat, held at $20.0 \pm 0.2°$ C.

The power supply for the spectrometer and the Helmholtz coils consists of a series of high capacity storage batteries.

With the present design the current is easily regulated and it is not difficult to hold it constant to well within 1 : 10^5, provided temperature constancy in the spectrometer coils is achieved.

The greatest current used at present, about 30 A, requires a battery voltage of 30 V. With small alterations of the cooling system and another current measuring device, it would be quite possible to attain β-energies of 1 MeV (corresponding to about 60 A).

TEMPERATURE CONTROL OF THE COIL SYSTEM

If the coil system gives a field B at a radius r at given coil temperatures and current, a different field B' will be obtained if the coil temperatures are changed by amounts ΔT_{outer} and ΔT_{inner} resp. Neglecting higher order terms we get

$$B' = B_0\left[1 - \beta\left(\left(\frac{B_{0\,\text{inner}}}{B_0} + \frac{r_0}{B_0}\frac{dB_{\text{inner}}}{dr}\right)\Delta T_{\text{inner}} + \left(\frac{B_{0\,\text{outer}}}{B_0} + \frac{r_0}{B_0}\frac{dB_{\text{outer}}}{dr}\right)\Delta T_{\text{outer}}\right)\right], \quad (4)$$

where β is the expansion coefficient of copper and $B_{0\,\text{inner}}$ and $B_{0\,\text{outer}}$ are the contributions from each coil to the total field.

If also the vacuum chamber experiences a temperature change $\Delta T_{\text{chamber}}$, we get a new r_0-value $r_0' = r_0(1 + \beta \Delta T_{\text{chamber}})$. The field at r_0' is $B_0' = B' + r_0 \beta \Delta T_{\text{chamber}} \dfrac{dB'}{dr}$.

Inserting the numerical values we get

$$\frac{B_0' r_0'}{B_0 r_0} = 1 - 1.6\ 10^{-5}[0.7\,\Delta T_{\text{outer}} - 0.2\,\Delta T_{\text{inner}} - 0.5\,\Delta T_{\text{chamber}}]. \quad (5)$$

From this it is found that a temperature change in one of the components of 1° C will cause an error in the $B\varrho$-determination of about $1 : 10^5$. It is to be noted that an overall temperature change will have no effect except for an insignificant change of the aberration properties.

For that reason, the inlets to the cooling coils were supplied with flow meters and the necessary water flow for maintaining a spectrometer temperature of about 18° C at different currents was determined. The temperature was measured by thermometers placed at the outside of each coil and on the chamber-lid. Furthermore a fan was placed over the spectrograph, as the vacuum chamber had a small tendency to warm up during the runs, due to the heat from the diffusion pump. It was found that temperature constancy within 0.2° C was attained after about 30 min.

ADJUSTMENT OF THE SPECTROMETER

After a careful geometrical setting of the spectrometer, the further adjustments were made with the aid of ThB sources. At first the median plane of the vacuum chamber was made to coincide with the symmetry plane of the coils. This was done with the aid of photographic records of the ThB F-line with the source position carefully adjusted. As a point at δz in the source plane will give an image at $-\delta z$, it was easily found from the axial position of the image how much the coil system had to be moved. The adjustment in radial direction is not very critical, especially if β is close to 3/8, so the geometrical adjustment, made with a accuracy of about $\pm$ 0.5 mm of the radius, was sufficient.

With a correct radial and axial setting, the focusing properties of the system were investigated. The axial focusing angle ϕ_z was determined by photographic records of the F-line from a source of well defined height, made at different values of ϕ. Evidently the image height will be the same as the source height at $\phi = \phi_z$ if higher order abberations are neglected. From these measurements it was found that $\phi_z = \sqrt{2}\pi + 0.10$. As is easily found, $\alpha = -\frac{1}{2}(1-\varepsilon)$ will give

$$\phi_z = \sqrt{2}\pi \sqrt{\frac{1}{1-\varepsilon}} \approx \sqrt{2}\pi \left(1 + \frac{\varepsilon}{2}\right)$$

and

$$\phi_r = \sqrt{2}\pi \sqrt{\frac{1}{1+\varepsilon}} \approx \sqrt{2}\pi \left(1 - \frac{\varepsilon}{2}\right).$$

The measurements thus indicate $\varepsilon = 0.045$.

As the radial focusing angle ϕ_r is of the greatest interest, it was decided to make a direct determination of it. In this case it is difficult to get a reasonable accuracy with the method used above for ϕ_z. Instead use was made of the aberration properties. If two rays are emanating from a point source with the angles γ_r' and $-\gamma_r'$, they will both hit the plane ϕ_r in $r^* = r_0 - \frac{4}{3} r_0 \gamma_r'^2$. When they hit a plane $\phi_r + \Delta\phi$, however, the first will fall on $r^{*\prime} = r_0 - \frac{4}{3} r_0 \gamma_r'^2 - r_0 \gamma_r' \Delta\phi$ and the second (in first order approximation) on $r^{*\prime\prime} = r_0 - \frac{4}{3} r_0 \gamma_r'^2 + r_0 \gamma_r' \Delta\phi$. When using GM-detector the appropriate formula is

$$\frac{\Delta(B\varrho)}{B\varrho} = -\frac{\gamma_r'^2}{3} - \frac{\gamma_r' \Delta\phi}{4}.$$

From this it is seen, that if $\Delta B\varrho / B\varrho$ is plotted as a function of γ_r', for a fixed $\Delta\phi$-value, a parabola is obtained which, in the case $\Delta\phi = 0$, i e $\phi = \phi_r$, is symmetric with respect to $\gamma_r' = 0$. The determination was made with the ThB F-line. A beam of narrow radial aperture $\Delta\phi$ was let through at different γ_r'- and $\Delta\phi$-values. The result is shown in fig. 9 in which the curve $\Delta I/I = -\frac{1}{3}\gamma_r'^2$ is also plotted. From this investigation it was found that ϕ_r was fairly close to $\sqrt{2}\pi - 0.11$, giving $\varepsilon = 0.049$ in good agreement with the ϕ_z-determination. In fig. 9 are also plotted the half widths of the lines at $\phi = \sqrt{2}\pi - 0.106$. From this curve it is found that the half width is rather large when γ_r' is high. This indicates a deviation from the ideal field form at high r-values, not detected in the investigations of the field form.

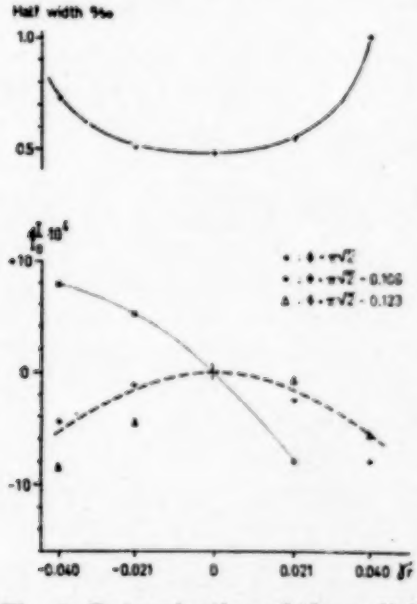

Fig. 9. Determination of the radial focusing angle (cf. text)

An α-value $-\frac{1}{2}(1-\varepsilon)$ causes an astigmatism $\delta_{as} = r_0 \gamma_r \varepsilon \sqrt{2}\pi$ in the plane $\phi = \sqrt{2}\pi$. In the present case this means $\delta_{as} \approx 67\gamma_r$ which of course is quite unpermissible. It was necessary either to change the α-value or to place the detector in the experimentally determined ϕ_r-position and permit a first order astigmatism in axial direction of the magnitude $2r_0\gamma_z\varepsilon\sqrt{2}\pi$. The

second alternative had not any serious drawbacks. The increased image height would cause an intensity loss, but this could, if necessary, easily be compensated for by increasing the slit height. The dispersion would be $4r_0(1-\varepsilon)\,\Delta B\varrho/B\varrho$ instead of $4r_0\,\Delta B\varrho/B\varrho$, a decrease of 5 %, which is unimportant. Furthermore, there would be insignificant modifications of the order of magnitude ε in the second order coefficients of (2). It was thus decided to retain the α-value and use a source-detector angle $\phi = \phi_r = \sqrt{2}\pi - 0.11$.†

4. Some Preliminary Measurements. Comparison with Theoretical Resolution

In order to make a comparison with the theoretical resolution and to test the reliability of the instrument, some series of GM-measurements were made on the ThB F-and I-lines with different

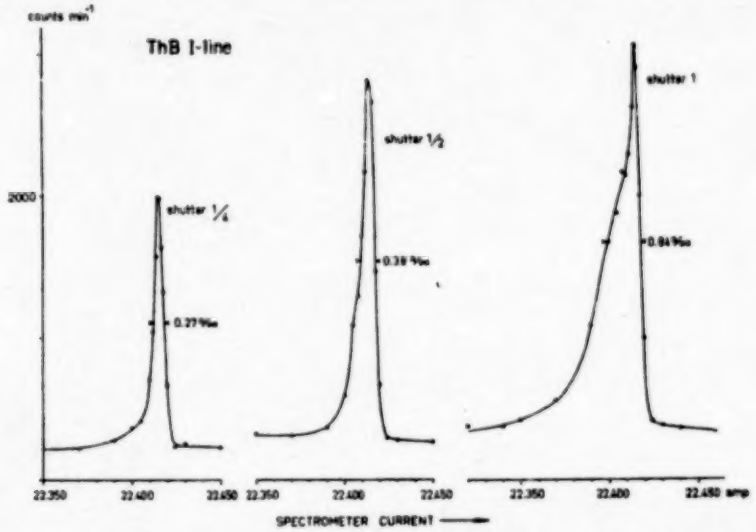

Fig. 10. Records of the ThB-I-line at different radial apertures (cf. table 2)

radial apertures and GM-slits. An abstract of the results of one series is given in table 2 and some line profiles in fig. 10.

As regards the theoretical resolution, this is in the present case (if the effects due to ε and the changed focusing angle are neglected)

$$\frac{\Delta(B\varrho)}{B\varrho} = \frac{a+s}{4r_0} + \frac{\gamma_r^2}{3}, \qquad (6)$$

† A recently performed slight change in α has completely eliminated this small astigmatic error.

where a is the source width and s the slit width. If run no. 1 (table) is taken as an example, it is found that the contribution from a and s will be about $0.25\,‰$ and that from γ_r, $0.10\,‰$, a total of $0.35\,‰$. If the half width is assumed to be half the base spread, the former will thus be $0.18\,‰$. The experimental value is $0.27\,‰$. Thus there is $0.09\,‰$ (= 30 eV) left for natural line width and remaining aberration effects. From this it is evident that the effect due to γ_s must be very small, i.e. β must be very close to 3/8.

TABLE 2

Line widths of ThB I-line at different resolution (cf. fig. 10)

Run	Radial aperture (radians)	Transmission ‰	Half Width ‰
1	0.015	0.5	0.27
2	0.031	1.0	0.38
3	0.062	2.0	0.84

Axial aperture 0.098 radians
Source dimensions 6 × 0.1 mm
GM - slit 0.2 mm

These measurements showed that it ought to be possible to use a considerably greater axial aperture without a great sacrifice of resolution. In the future it is intended to build a vacuum chamber permitting an axial aperture of about 0.3 radians, thus giving a maximum solid angle of 0.6 %. A still larger aperture will probably seriously affect the resolution because of deviations from the ideal field form at great distances from the symmetry plane and also higher order aberration effects.

Some determinations of the dispersion were also made. Two I-lines were exposed on the same film, with the spectrometer currents so adjusted that the distance between them was about 1 mm. The dispersion was in good agreement with the theoretical value $1200 \times 0.951\, \Delta I/I$.

THE NATURAL LINE WIDTHS OF ThB CONVERSION LINES

In fig. 11 is shown GM-records of the ThB F, I, I_a and J-lines taken at high resolution and with a very small GM-slit. The difference in line width between the F (K-conversion) and the I, I_a

and J (L_I, L_{II} and M_I-conversion) is quite evident. (The difference is still more striking in fig. 12 and on the photographic record,

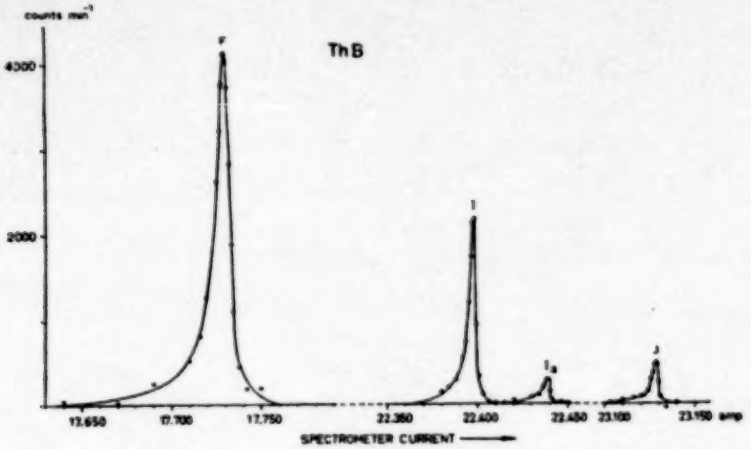

Fig. 11. Records of ThB-lines (cf, table 3)

fig. 13.) In table 3 the corrected half widths are given. The result confirms the results of Slätis[3]) made by photographic determination in that the difference between K- and L_I-width is about 60 eV.

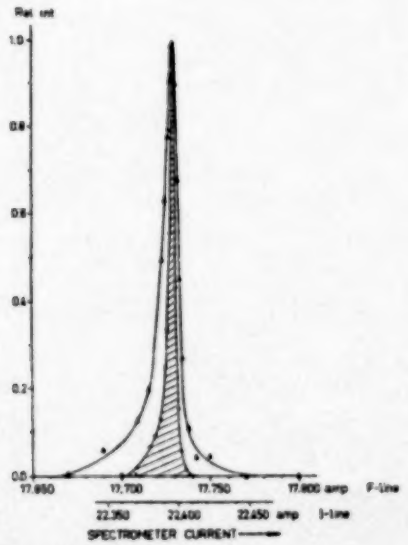

Fig. 12. Comparison of ThB F- and I-lines taken at identical conditions

As the lines are asymmetrical, which might be due to some small straggling effects in the source, there is good reason to calculate

the half widths only from the high energy part of the lines. These half widths are also given in the table.

TABLE 3
Line widths of ThB - conversion lines

Line	Half width			Corrected half width *		
	‰	eV	Difference	‰	eV	Difference
F	0.50	126	—	0.40	101	—
I	0.19	71	55	0.13	49	52
I_a	(0.24	91	35) **	0.13	49	52
J	0.20	79	47	0.13	51	50

Source dimensions: 8 × 0.05 mm
Radial aperture: 0.01 radians

* Two times the difference between half value on high energy slope and peak value.
** Probably some overlapping from the I-line.

The natural widths of the L- and M-lines are expected to be about 10 eV i.e. 0.025 ‰. It might be possible to obtain this resolution in the future with the present instrument.

ENERGY DETERMINATION OF SOME ThB AND ThC LINES

As the stronger lines in ThB-deposits are commonly used as reference-lines, a precision determination of some of them was made. This was at the same time a check of the instrument.

The method used was to make precision determinations of the current ratio between conversion lines belonging to the same γ-transition and then to use the energy difference between the lines, found from X-ray data, to calculate the absolute $B\varrho$-values. The formulas are:

$$(B\varrho)_I^2 = A + \sqrt{A^2 + B}, \tag{7}$$

$$A = \frac{\alpha^2 + 1}{(\alpha^2 - 1)^2} \cdot \tilde{\nu}^2 \left(\frac{h}{e}\right)^2, \tag{8}$$

$$B = \frac{4 - s^2}{(\alpha^2 - 1)^2} \tilde{\nu}^2 \left(\frac{h}{e}\right)^2 c^2 \left(\frac{m}{e}\right)^2, \tag{9}$$

$$\alpha = \frac{(B\varrho)_{II}}{(B\varrho)_I}, \tag{10}$$

$$s = \left(\frac{h}{mc}\right)^2 \tilde{\nu}^2. \tag{11}$$

The values used for the fundamental constants were taken from DuMond and Cohen [4]).

The following formulas are useful for the error calculation:

$$\frac{\Delta(B\varrho)}{B\varrho} = \frac{A}{\sqrt{A^2+B}} \cdot \frac{\Delta A}{A} + \frac{1}{4(B\varrho)^2} \cdot \frac{B}{\sqrt{A^2+B}} \frac{\Delta B}{B}, \quad (12)$$

$$\frac{\Delta A}{A} = \left(\frac{2\alpha^2}{\alpha^2+1} - \frac{4\alpha^2}{\alpha^2-1}\right)\frac{\Delta\alpha}{\alpha} + \frac{2\Delta\tilde{\nu}}{\tilde{\nu}} + \frac{2\Delta(h/e)}{h/e}, \quad (13)$$

$$\frac{\Delta B}{B} = -\frac{4\alpha^2}{\alpha^2-1}\frac{\Delta\alpha}{\alpha} + \frac{2\Delta\tilde{\nu}}{\tilde{\nu}} + \frac{2\Delta(h/e)}{h/e} + \frac{2\Delta(m/e)}{m/e} + \frac{2\Delta c}{c}. \quad (14)$$

I-F *ratio*

The first line-pair to be investigated was the F and I lines. A preliminary investigation was made, in which GM-records of the lines were taken at several source dimensions and apertures. It was then found that the ratio between the peak values varied with the resolution, so that a moderate resolution resulted in a distinctly lower I/F ratio than a high one. It was thus necessary to work at the highest possible resolution. In that case the difference in line

TABLE 4

I-F and I_a-F Ratios

Run	I-F	I_a-F	Run	I-F	I_a-F
1	1.26323		11	1.26323	1.26553
2	22	1.26550	12	24	50
3	23	51	13	26	50
4	24		14	21	53
5	23	49	15	19	
6	23	50	16	26	53
7	22	51	17	22	50
8	25		18	21	
9	23	46	19	19	
10	25		20	26	

Mean values: I-F : 1.26323 ± 0.000015
I_a-F : 1.26550 ± 0.000015
Source dimensions 8 × 0.05 mm
Radial aperture: 0.01 radians

width between the F and I lines made a comparision of the peak values from GM-records rather unreliable (unless a careful folding process was made), as the smallest GM-slit available (0.05 mm)

was of the same order of magnitude as the line widths. In that situation it seemed more convenient to use the photographic method. Two F-lines and two I-lines were exposed on the same film with the current values so adjusted that the distance between the lines was about 1 mm. With the lines in the interval $r_0 \pm 2$ mm, the dispersion was linear within the accuracy of the measurements and no broadening of the outer lines was observed. The distance between the lines was measured on a micro-comparator with an accuracy of $\pm$ 0.01 mm and sometimes also on micro-photometer records of the films. The current ratio was then calculated by linear interpolation. The results of 20 measurements are given in table 4. The energy difference between the F and I lines corresponds to the Bi K-L_I transition. As this X-ray line is forbidden and thus very

TABLE 5

Calculation of $B\varrho$-values from I-F ratio

X-ray transition	$\tilde{\nu}/R$	
Bi K — L_I =		
$K\beta_3$ — $L\beta_4$	6395.59 — 934.72	= 5460.87
$K\beta_1$ — $L\beta_3$	6433.02 — 972.91	= 5460.11
$K\alpha_1$ + $L\alpha_1$ — $L\beta_9$	5679.36 + 798.28 — 1016.97	= 5460.67

Mean value: 5460.55 $\pm$ 0.44

I-F ratio: 1.26323 $\pm$ 0.000015
$(B\varrho)_F$ = 1388.44 $\pm$ 0.10
$(B\varrho)_I$ = 1753.91 $\pm$ 0.14

weak, a mean value of the combinations $K\beta_3$—$L\alpha_4$, $K\beta_1$—$L\beta_3$ and $K\alpha_1 + L\alpha_1 - L\beta_9$ was used. The data used in the calculation are given in table 5. The values found are $B\varrho_F = 1388.44 \pm 0.10$ and $B\varrho_I = 1753.91 \pm 0.14$. (The X-ray data were taken from Cauchois and Hulubei [5]).)

I_a-F *ratio*

On some of the I-F films it was also possible to measure the positions of the I_a-lines from which the I_a-F ratio could be calculated (cf table 4). This line pair corresponds to Bi K—L_{II} i.e. Bi $K\alpha_2$. In table 6 are given the values used in the calculation. The result was $B\varrho_F = 1388.45 \pm 0.10$ and $B\varrho_{I_a} = 1757.07 \pm 0.14$.

TABLE 6

Calculation of $B\varrho$-values from I_a-F ratio

X-ray transition	$\tilde{\nu}/R$
Bi K—L_{II} = $K\alpha_2$	5510.49 ± 0.30 *

I_a-F -ratio : 1.26550 ± 0.000015
$(B\varrho)_F$ = 1388.44 ± 0.10
$(B\varrho)_{I_a}$ = 1757.07 ± 0.14

* estimated error

J-F *ratio*

As the I and J-lines seem to be of about the same shape, it was considered permissible to determine this current ratio from GM-records. The J-I ratio is unfavourable for $B\varrho$-calculations as the lines are too close to each other and for that reason, the J-F ratio was calculated with the aid of the previously determined

TABLE 7

Calculation of $B\varrho$-values from J-F ratio

X-ray transition	$\tilde{\nu}/R$
Bi K — M_I =	
$K\alpha_2 + n$	5510.49 + 862.60 = 6373.09
$K\alpha_1 + l$	5679.36 + 693.80 = 6373.16

Mean value: 6373.13 ± 0.04

J-F ratio:
$(B\varrho)_F$ = 1388.45 ± 0.10
$(B\varrho)_J$ = 1811.11 ± 0.15

I-F ratio. In this case also, the corresponding X-ray transition (K—M_I) is forbidden, and a mean value of $K\alpha_2 + n$ and $K\alpha_1 + l$ was used. The imput values are given in table 7. The result was $B\varrho_F$ = 1388.45 ± 0.10 and $B\varrho_J$ = 1811.11 ± 0.14.

B-A *ratio*

Finally, photographic records of the A, B and I lines were taken in the same manner as for the I-F determination (table 8). From the B-A ratio and the corresponding X-ray transitions (Tl L_I—M_I) $L\beta_9 - L\alpha_1 + l$ and $L\beta_{10} - L\beta_1 + \eta$, the $B\varrho$-values were cal-

TABLE 8

I-B and B-A ratios

Run	I-B	B-A
1	2.68831	1.22120
2	54	25
3	78	27
4	79	23
5	47	24

Mean values: I-B : 2.6886 ± 0.0001
B-A : 1.22124 ± 0.00002

culated (table 9). The values found were $B\varrho_A = 534.20 \pm 0.03$ and $B\varrho_B = 652.38 \pm 0.04$. The I-line was, in this connection, used as a reference line for the mc/h-determination.

TABLE 9

Calculation of $B\varrho$-values from B-A Ratio

X-ray transition	$\tilde{\nu}/R$
Tl $L_1 - M_1$ =	
$L\beta_9 - L\alpha_1 + l$	954.35 — 756.29 + 659.39 = 857.45
$L\beta_{10} - L\beta_1 + \eta$	974.27 — 899.45 + 809.72 = 857.54

Mean value: 857.50 ± 0.05

B-A ratio: 1.22124 ± 0.00002
$(B\varrho)_A = 534.21 \pm 0.03$
$(B\varrho)_B = 652.40 \pm 0.04$

Determination of mc/h

When the $B\varrho$-ratios of two pairs of conversion lines and the ratio between the lines are known, it is possible to eliminate all fundamental constants from eqs (7)-(9) except mc/h. A calculation gives

$$mc/h = \frac{2\gamma}{b_2\gamma^4 - b_1}\sqrt{(a_1 - a_2\gamma^2)(a_1 b_2\gamma^2 - a_2 b_1)}, \tag{15}$$

where

$$a_i = \frac{\alpha_i^2 + 1}{(\alpha_i^2 - 1)^2}\tilde{\nu}_i^2, \tag{16}$$

$$b_i = \frac{4 - s_i^2}{(\alpha_i^2 - 1)^2} \cdot \tilde{\nu}_i^2, \tag{17}$$

$$\gamma = (B\varrho)_{11}/(B\varrho)_{12}. \tag{18}$$

The subscripts refer to each line-pair.

This equation gives a possibility of determing mc/h. (In eq. (17) mc/h is contained in the s^2-term, but as $s^2 \ll 4$, this will not affect the calculations.)

Using the pairs I-F and B-A, a calculation of mc/h was performed.

5. Error Discussion

When the appropriate numerical values for the I-F case are inserted in eq. (12), the following expression is obtained:

$$\frac{\Delta B\varrho}{B\varrho} = -3.3\frac{\Delta\alpha}{\alpha} + 0.65\left(\frac{\Delta\tilde{\nu}}{\tilde{\nu}} + \frac{\Delta h/e}{h/e}\right) + 0.35\left(\frac{\Delta m/e}{mc/h} + \frac{\Delta c}{c}\right).$$

Assuming the following probable errors:

$$\frac{\Delta\alpha}{\alpha} = 1.2\cdot 10^{-5}, \qquad \frac{\Delta h/e}{h/e} = 3.6\cdot 10^{-5},$$

$$\frac{\Delta\tilde{\nu}}{\tilde{\nu}} = 8\cdot 10^{-5}, \qquad \frac{\Delta m/e}{m/e} = 2.8\cdot 10^{-5}, \qquad \frac{\Delta c}{c} = 0.27\cdot 10^{-5}$$

(the probable errors for the fundamental constants are taken from DuMond and Cohen), it is found that the probable error in $B\varrho$ is $7\cdot 10^{-5}$. The contribution from the error in α alone is $4\cdot 10^{-5}$ and from $\tilde{\nu}$, $5\cdot 10^{-5}$. As regards systematic errors, there seems to be no reason to suspect any of the same order of magnitude.

The probable errors for the other line-pairs have been computed in the same way, where the errors given in tables 6—9 have been used.

The three line pairs I-F, I_a-F, J-F give three values of $(B\varrho)_F$ which are partly dependent on each other, as the same F-line position is used in the calculations and furthermore some of the X-ray lines used are common for two pairs. For that reason, an error calculation, where use is made of the total information obtained about the $(B\varrho)_F$-value, would necessitate the introduction of rather arbitrary values of the errors in F-line position and X-ray energies. It seems sufficient to say that the very good consistency between the three lines indicates that the error limits given are rather conservative.

Differentiation of eq. (15) and introduction of the actual numerical

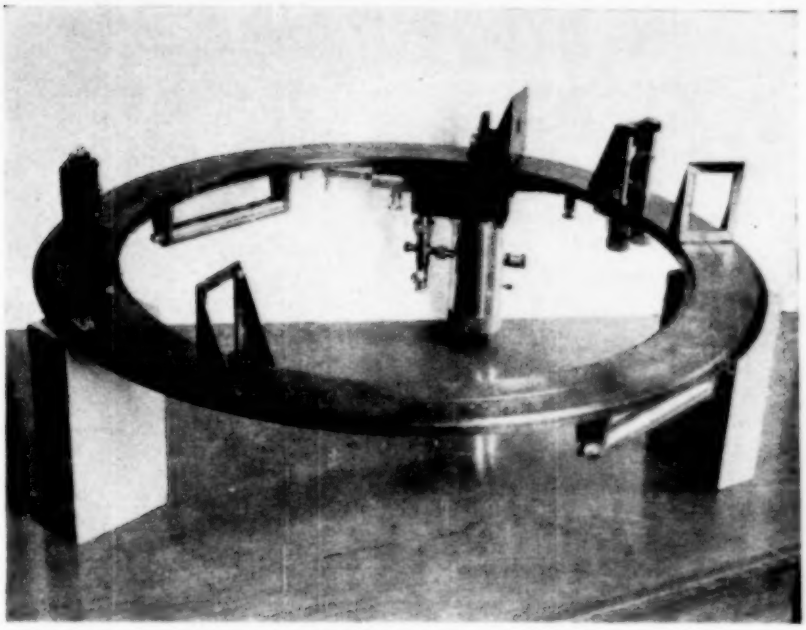

Fig. 6. Lid of the vacuum chamber, supporting sample holder with lock device (in the background), shutters GM-counter and handles

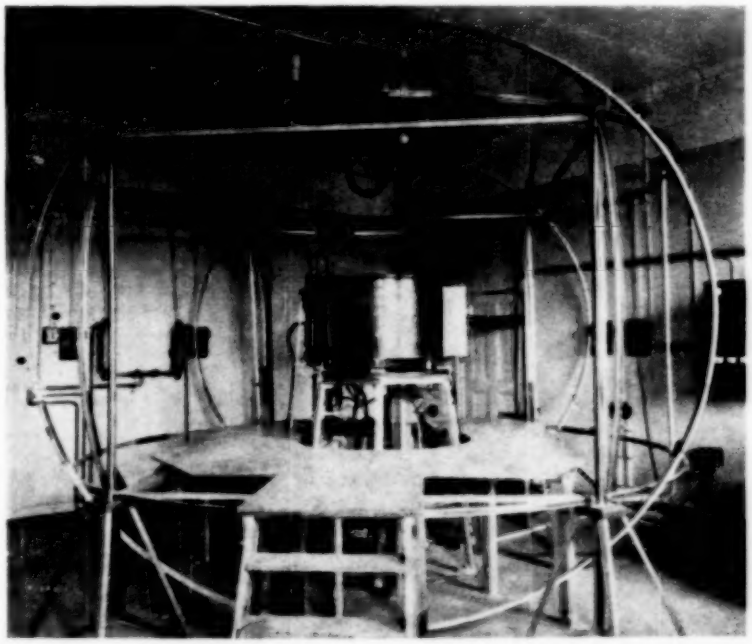

Fig. 7. View of the spectrometer with Helmholtz system

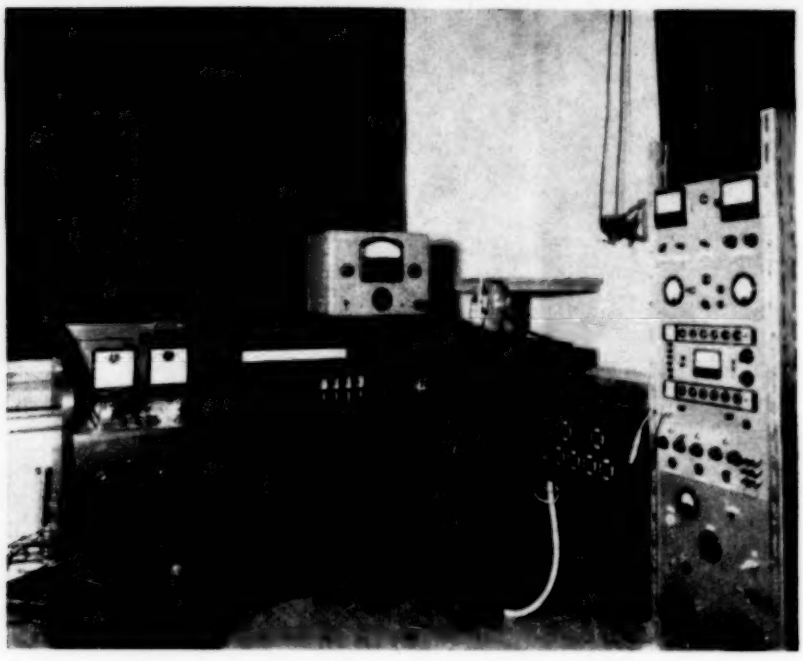

Fig. 7a. View of the measuring equipment, showing compensator table with magnetometer and rack with vacuum measuring instrument, scaler and Helmholtz field controls

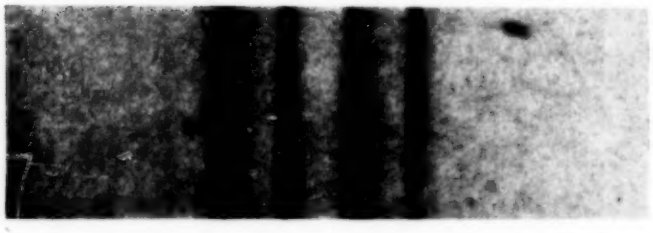

Fig. 13. Photographic records of two F- and two I-lines showing the different line widths of K_I- and L_I-conversion lines (7×)

values, give the following expression for the error in h/mc:

$$\frac{\Delta h/mc}{h/mc} = 27\frac{\Delta\alpha_{IF}}{\alpha_{IF}} - 5.3\frac{\Delta\tilde{\nu}_{IF}}{\tilde{\nu}_{IF}} - 26\frac{\Delta\alpha_{BA}}{\alpha_{BA}} + 4.3\frac{\Delta\nu_{BA}}{\nu_{BF}} - 0.76\frac{\Delta\gamma}{\gamma}.$$

Introduction of the appropriate errors gives a relative probable error of $7 \cdot 10^{-4}$.

TABLE 10

Comparison between $B\varrho$-values of A and B lines obtained from X-ray data and from I-A and I-B ratios

	I-A	I-B	B-A
Ratio	3.28334	2.68858	1.22124
$B\varrho_A$	534.18 *	—	534.21
$B\varrho_B$	—	652.36 *	652.40

* Assuming $(B\varrho)_I = 1753.91$

As a test of the linearity of the spectrometer, the $B\varrho$-values for the A- and B-lines calculated from the B-A ratio were compared with the values obtained from the I-A and I-B ratios when $(B\varrho)_I = 1753.91$. The result is given in table 10. The agreement is very good.

6. Comparison with Earlier Determinations

F- *and* I-*Lines*

Precision determinations of these lines have lately been undertaken by Lindström[6]), Craig and Dietrich[7]) and Meyer and Schmidt[8]).

Lindström made a determination of the $B\varrho$-ratio using a semicircular spectrometer with GM-detector. He obtained half-widths of about 1.4 and 0.8 ‰ resp. for the lines. The field was determined with an accuracy better than $2 : 10^5$. The line positions were however determined from the intersection of the high energy slope and the background. As this slope actually is steeper for the I-line than for the F-line, this method will give a too low $B\varrho$-ratio (apart from the effect due to the moderate resolution). His value 1.26317 ± 0.00002 is also lower than that found in the present investigation.

Craig and Dietrich have made a determination of the $B\varrho$-ratio by studying the angular rotation of a line image in a long lens

spectrometer and found the value 1.26317 ± 0.00012, thus in exact agreement with Lindström, although with greater error limits.

Meyer and Schmidt have determined the F-*line* by comparing it with electrons, accelerated by an accurately determined potential. They used an ironfree lens spectrometer and obtained half widths of about 1.3 ‰ for the F-line and 1.1 ‰ for the electrons. The $B\varrho$-value obtained was 1388.56 ± 0.21.

TABLE 11

Comparison with earlier determinations

Reference	F	I	I_a	J
Ellis	1385.8 ±3.0	1751.0 ±2	1754.0 ±2	1807.7 ±2
Surugue	1388 ±1.4	1752 ±2	1755 ±2	1809 ±2
Lindström	1388.55±0.20	1754.01±0.25	—	—
Craig & Dietrich	1388.52±0.30	1753.94±0.39	—	—
Meyer&Schmidt	1388.56±0.21	—	—	—
Present work	1388.45±0.10	1753.91±0.14	1757.07±0.14	1811.11±0.15

	A	B
Ellis	536.50±0.5	652.57±0.7
Surugue	534 ±0.5	652 ±0.7
Craig & Dietrich	533.66±0.12	—
Present work	534.21±0.03	652.40±0.04

The results of these determinations and the present one is abstracted in table 11. The agreement between the earlier determinations is very good, whereas the present one gives a slightly lower value. This is probably due to the higher resolution used in the present case.

I_a-, J-, A- *and* B-*Lines*

No determinations of these lines have been reported since the work of Ellis [9,10]), Surugue[11]) and others in the 1930's, with the exception of Craig and Dietrich, who have determined the A-line from the F-A-ratio. A comparison is made in table 11, from which it is seen that the agreement is often rather poor.

h/mc Determination

The most accurate determination of h/mc is obtained by DuMond *et al.*[12]) from annihilation radiation data. They report $h/mc = (2.4262 \pm 0.0003)10^{-10}$ cm. The present determination gives $(2.4262 \pm 0.0016)10^{-10}$, thus in exact agreement although with a greater probable error.

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RELATIVE CONVERSION PROBABILITIES IN THE K, L_I AND L_{II} SHELLS OF THE 238,6 keV γ-LINE OF Th B.

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Accurate calculations of conversion coefficients for electric as well as magnetic radiation of different multipole orders have been performed by Rose *et al.* [1]) The question has arisen [2]) whether the finite size of the nucleus, neglected in these calculations, might lead to any appreciable errors. Thus, for instance, it was found by Nijgh [3]) that the K conversion coefficient, the K/L ratio and $L_I/L_{II}/L_{III}$ ratios of the 279 keV γ-transition in Hg^{203} obtained experimentally were not reconcilable unless one introduced a correction factor of about 0,7 for the theoretical K and L_I conversion coefficients for M1 transitions. No corrections were applied to the L_I and L_{III} conversion coefficients

In order to examine this problem somewhat more closely a precise measurement has been performed of the relative intensities of the three conversion lines of the 238,6 keV γ-line in ThC that originate in the K, L_I and L_{II} subshells. These are the wellknown F, I and Ia lines (Ellis' notation). Although the energies of these lines are known to a considerable degree of accuracy only crude measurements of their intensities have been made so far. In those cases where the intensities have been studied by direct counting methods the resolution has not been sufficient to resolve the L components, whereas in the cases where these lines have been resolved only photographic registration has been employed, giving results of poor accuracy. In the present investigation an iron-free double-focusing spectrometer [4]) has been used together with a GM counter. The resolution was set to be 0.35 $^0/_{00}$(at the I line). The GM mylar window had a thickness of 600 $\mu g/cm^2$, which is sufficient to render unnecessary any corrections for absorption in this energy region. The source was a ThB deposit, obtained from a RdTh source by electric collection on a 0.1×8 mm Ag wire. Care was taken, that only one side of the wire was activated.

Two sets of measurements were performed, using two different samples. Fig. 1 shows one set of curves. Table 1 includes the intensity ratios K/L_I and L_I/L_{II} obtained experimentally as well as the corresponding ratios obtained theoretically by interpolation for $Z = 83$ and an energy of 238 keV, using Rose's tables for M1 transitions.

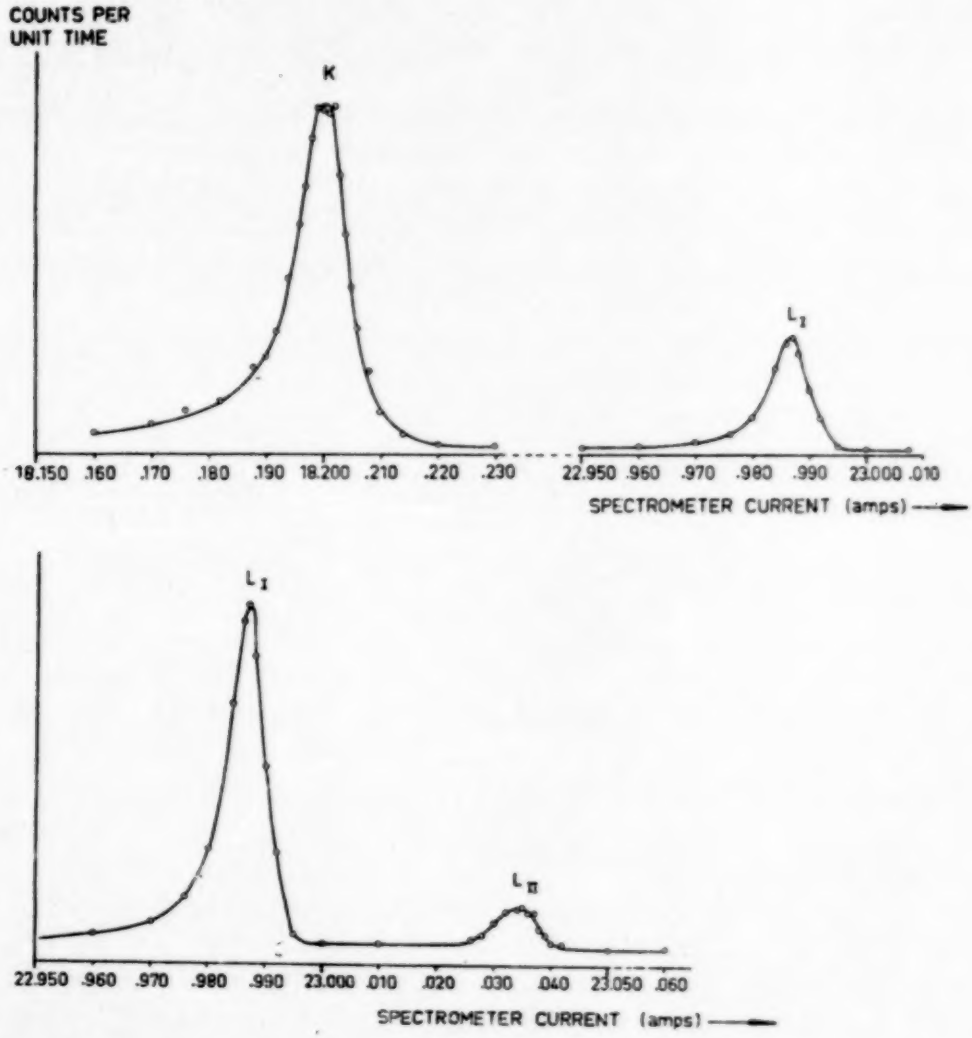

Fig. 1

TABLE I

Conversion ratio	Run 1	Run 2	Theoretical for M1
K/L_I	5.8	6.0	6.35 ± 0.2
L_I/L_{II}	9.3	10.6	10.6 ± 0.6

We estimate the error in our intensity measurements to be $< 7\ \%$. From table 1 it is evident that *both* intensity ratios lie well within the errors of the theoretical ones for pure M1 transitions. It

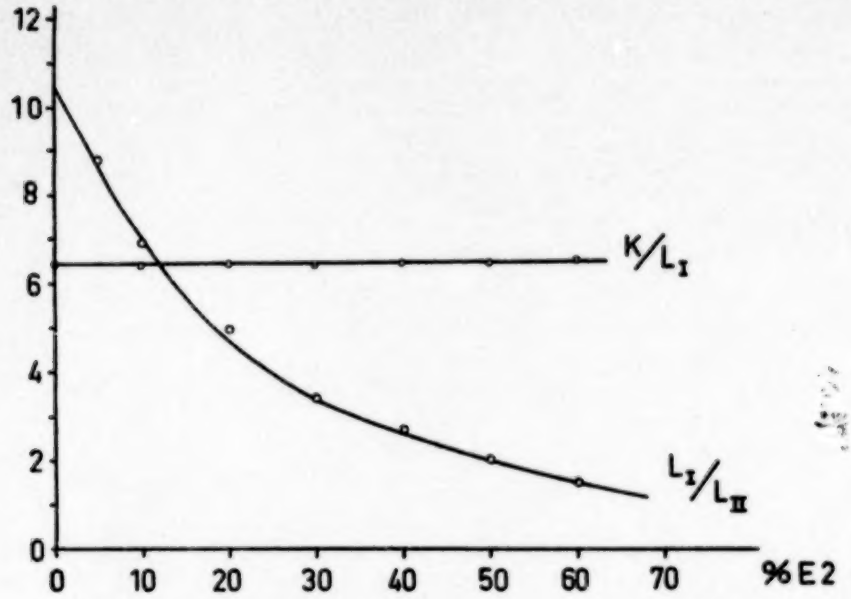

Fig. 2

is interesting to note that Feather and Richardson [5]) were inclined to draw the conclusion from the systematics of β transitions that the 238 keV transition is associated with a spin change from 1 to 0 with no parity change, which, according to selection rules, makes it a pure magnetic dipole transition. More recently Martin and Richardson [6]) found the K conversion coefficient for the 238 keV transition to be considerably lower than given by theory, which leaves open the possibilities that the magnetic dipole assignment is incorrect *or* that Rose's K conversion coefficients should be subject to an adjustment.

All evidence seems to speak against the former alternative. Let us consider the latter: Nijgh's finding in the case of Hg^{203} that the nuclear finite size seems to affect the K and L_I conversion coefficients to about the same extent would explain the agreement between our experimental K/L_I ratio and that obtained from the tables of Rose. The fact that also our L_I/L_{II} ratio is in very good agreement with the uncorrected theoretical value is more difficult to explain on these grounds, since the effect on L_I conversion, involving a $2s$ electron, is expected to be greater than on L_{II} conversion, involving a $2p$ electron. (Nijgh's results support these speculations, since he

obtained consistency without correcting the L_{II} conversion coefficients). The agreement with Rose's L_I/L_{II} ratio cannot be explained by an E2 admixture, since such an admixture would tend to lower the L_I/L_{II} ratio as would the finite size effect. Within our limits of error we can set the upper limit of E2 mixing to be 3 %.

Our results indicate that the relative conversion probabilities for a magnetic dipole transition, at least in this case, are well described by the theoretical results of Rose *et al.* If the finite nuclear size effect does have an influence on the conversion prohabilities one would be inclined to draw the conclusion that the necessary corrections are of the same order of magnitude for the K, L_I and L_{II} shells. Since, according to the above considerations, this seems somewhat unlikely it would be of great interest to check these conclusions by a direct absolute measurement of the conversion probabilities involved.

From the viewpoint of the shell model the two odd nucleons in the ground state of $_{83}Bi^{212}$ would be expected to be in the $g_{9/2}$ and $h_{9/2}$ states. According to Nordheim's second rule the odd proton and neutron couple to give a spin = 0 and odd parity. Our measurements then indicate odd parity and spin 1 for the 238 keV excited state in $_{83}Bi^{212}$. Our finding that the transition is a *pure* magnetic dipole with no E2 mixing stronlgy supports these spin assignments since any E2 mixing is in that case prohibited according to selection rules.

We are glad to acknowledge the continual cooperation and interest of Dr. A. H. Wapstra in this work.

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PRINCIPAL AXIS TRANSFORMATION FOR A FREE NUCLEON COUPLED TO AN UNCHARGED SCALAR MESON FIELD

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Abstract: The Hamiltonian for a free nucleon coupled to an uncharged scalar meson field is transformed to normal modes after dropping all terms higher than quadratic in the (field and particle) variables. The resulting normal modes represent scattering states and the "runaway solutions" well-known from the corresponding electromagnetic case, whereas no excited states of the nucleon occur.

1. Introduction

Some time ago van Kampen [1]) showed that the non-relativistic Hamiltonian for a free or elastically bound electron coupled to the electromagnetic field can be explicitly transformed to normal modes. The essential possibility of such a transformation rests with the fact that the nonrelativistic (or dipole) Hamiltonian is a quadratic form in all (field and particle) variables. In a recent paper it was shown by one of the authors [2]) that van Kampen's method can be formulated in such a way that the transformation formulae are obtained in a straightforward manner resulting in a general method for treating quadratic Hamiltonians. Especially in the case of large coupling constants it is difficult, if not impossible, to get reliable results by the usual perturbation theory, and it was hoped that it might be possible to treat these cases with the van Kampen method. It will be shown in this paper that the simple case of a nonrelativistic nucleon coupled to a neutral scalar meson field † can indeed be handled very easily, in spite of the finite rest mass of the meson

† There is, of course, no essential difficulty in treating this case by the original van Kampen method (cf. McVoy, Diplom-Arbeit, Göttingen), but then the results partially appear in the form of infinite series which are difficult to sum.

which at first sight might have led to serious difficulties because of retardation effects†.

2. Derivation of the Quadratic Hamiltonian

We start with the Hamiltonian ††

$$H = \frac{\mathbf{P}^2}{2m} + g\varphi(\mathbf{R}) + \frac{1}{8\pi}\int \{(4\pi\psi)^2 + (\operatorname{grad}\varphi)^2 + \mu^2\varphi^2\}\, d\tau, \quad (1)$$

where $\mathbf{P}$ and $\mathbf{R}$ are momentum and position vector of the nucleon, m its unrenormalized mass, $\varphi(\mathbf{r}, t)$ and $\psi = \dot{\varphi}/4\pi$ the meson field and its canonically conjugate field, μ the Compton wave length of the meson and g the coupling constant. From (1) follow the well-known equations of motion

$$m\ddot{\mathbf{R}} = -g \operatorname{grad}\varphi$$
$$\Delta\varphi - \ddot{\varphi} - \mu^2\varphi = 4\pi g\delta(\mathbf{R} - \mathbf{r}).$$

For convenience we enclose the system nucleon + field into a large reflecting sphere with radius L, so that we have the boundary condition

$$\varphi(L) = \psi(L) = 0. \quad (2)$$

In order to get a quadratic Hamiltonian which can be treated with van Kampen's method, we have to use a Taylor-expansion of the coupling term:†††

$$g\varphi(\mathbf{R}) = g\varphi(0) + g(\mathbf{R}\cdot\operatorname{grad}\varphi)_0 + \frac{g}{2}\{\mathbf{R}\cdot\operatorname{grad}(\mathbf{R}\cdot\operatorname{grad}\varphi)\}_0, \quad (3)$$

where the index 0 means $\mathbf{r} = 0$. Furthermore we decompose φ into its even and odd parts:

$$\varphi = \varphi_1 + \varphi_2, \quad (4)$$

where $\varphi_1(-\mathbf{r}) = -\varphi_1(\mathbf{r})$, $\varphi_2(-\mathbf{r}) = \varphi_2(\mathbf{r})$.

The Taylor expansion (3) defines the nature of the approximation which limits the validity of the results. The 'normal vibrations' of the total system must lead to a space-time dependence of φ of the form

† The Hamiltonian (13) might have contained time integrals over the whole previous path of the nucleon because the velocity of mesons with non-zero rest-mass can have any value between 0 and c, cf. below after eq. (9).

†† We put c (velocity of light $) = 1$ (except in (5)). As the transformations involved are linear, we can treat the whole problem classically at first, and perform the quantization after the principal axis transformation.

††† It is to be noted that this expansion destroys the translational invariance of the Hamiltonian, but this should not bring any trouble as long as this approximation is valid at all. Thanks are due to Dr. G. Höhler, Göttingen, for calling our attention to this point.

$$\varphi \propto \exp(ikr - i\omega t)$$

with

$$k^2 - \omega^2/c^2 = -\mu^2.$$

The condition for the validity of the approximation involved is then

$$kR \ll 1, \text{ i.e. } \sqrt{\left(\frac{R\omega}{c}\right)^2 - (R\mu)^2} \ll 1, \tag{5}$$

which is unfortunately not fully identical with the nonrelativistic approximation, as it would be if $\mu = 0$.†

Therefore it is not quite easy to interpret the condition (5) physically, apart from the statement that the wave length of the considered component of the meson field must be large compared with the average value of the amplitude of vibration of the nucleon. Substituting (3) and (4) into the Hamiltonian, and taking into account the fact that the integration over the whole space of the product of an odd and an even function vanishes, we get

$$\begin{aligned} H = &\frac{\mathbf{P}^2}{2m} + g\varphi_2(0) + g(\mathbf{R}\cdot\operatorname{grad}\varphi_1)_0 \\ &+ \frac{1}{2}g\{\mathbf{R}\cdot\operatorname{grad}(\mathbf{R}\cdot\operatorname{grad}\varphi_2)\}_0 \\ &+ \frac{1}{8\pi}\int\{(4\pi\psi_1)^2 + (\operatorname{grad}\varphi_1)^2 + \mu^2\varphi_1^2\}\,d\tau \\ &+ \frac{1}{8\pi}\int\{(4\pi\psi_2)^2 + (\operatorname{grad}\varphi_2)^2 + \mu^2\varphi_2^2\}\,d\tau \end{aligned} \tag{6}$$

and the equations of motion

$$\begin{aligned} m\ddot{\mathbf{R}} &= -g\,(\operatorname{grad}\varphi_1)_0 + g\,\{\operatorname{grad}(\mathbf{R}\cdot\operatorname{grad}\varphi_2)\}_0 \\ \Delta\varphi_1 - \ddot{\varphi}_1 - \mu^2\varphi_1 &= -4\pi g\,\{\mathbf{R}\cdot\operatorname{grad}\delta(\mathbf{r})\} \\ \Delta\varphi_2 - \ddot{\varphi}_2 - \mu^2\varphi_2 &= 4\pi g\delta(\mathbf{r}) + 2\pi g\,\{\mathbf{R}\cdot\operatorname{grad}(\mathbf{R}\cdot\operatorname{grad}\delta(\mathbf{r}))\}. \end{aligned}$$

Then we make a further decomposition

$$\varphi_2 = \varphi_3 + \varphi_4, \tag{7}$$

$$\Delta\varphi_3 - \mu^2\varphi_3 = 4\pi g\,\delta(\mathbf{r}), \tag{8}$$

$$\Delta\varphi_4 - \ddot{\varphi}_4 - \mu^2\varphi_4 = 2\pi g(\mathbf{R}\cdot\operatorname{grad}(\mathbf{R}\cdot\operatorname{grad}\delta(\mathbf{r}))). \tag{9}$$

This gives

$$\varphi_3 = -g\frac{e^{-\mu r}}{r}$$

and we note that

† In this case the condition is simply $R\omega/c \approx v/c \ll 1$, where v is the velocity of the nucleon.

$$\psi_3 = 0 \text{ because } \dot{\varphi}_3 = 0$$

and that φ_4 and therefore ψ_4 are of the order R^2. As we are limiting ourselves to terms up to the order R^2 in the Hamiltonian, we can omit the terms $\propto \varphi_4^2$, ψ_4^2 and $R^2\,\varphi_4$. By performing a partial integration and using (8), we see that all further terms containing φ_4, namely

$$g\varphi_4(0) + \frac{1}{4\pi}\int \{\text{grad}\,\varphi_3 \cdot \text{grad}\,\varphi_4 + \mu^2\varphi_3\varphi_4\}\,d\tau,$$

cancel each other. This cancellation is indeed the reason why we do not get into difficulties with retardation effects which would otherwise arise from eq. (9).† So we are left with the Hamiltonian

$$\begin{aligned} H = {} & \frac{\mathbf{P}^2}{2m} + g\,(\mathbf{R}\cdot\text{grad}\,\varphi_1)_0 \\ & + \frac{1}{8\pi}\int \{(4\pi\psi_1)^2 + (\text{grad}\,\varphi_1)^2 + \mu^2\varphi_1^2\}\,d\tau \\ & + \frac{g}{2}\{\mathbf{R}\cdot\text{grad}\,(\mathbf{R}\cdot\text{grad}\,\varphi_3)\}_0 + g\varphi_3(0) \\ & + \frac{1}{8\pi}\int \{(\text{grad}\,\varphi_3)^2 + \mu^2\varphi_3^2\}\,d\tau. \end{aligned} \tag{10}$$

The last two terms represent the static self energy††

$$\begin{aligned} & g\varphi_3(0) + \frac{1}{8\pi}\int \{(\text{grad}\,\varphi_3)^2 + \mu^2\varphi_3^2\}\,d\tau \\ & = \tfrac{1}{2}\,g\varphi_3(0) = -\frac{g^2}{2}\lim_{r\to 0}\left(\frac{1}{r} - \mu\right) \end{aligned} \tag{11}$$

and will be omitted in the following as they have no influence on the motion of the particle or the field. Finally we get

$$\begin{aligned} & g\{\mathbf{R}\cdot\text{grad}\,(\mathbf{R}\cdot\text{grad}\,\varphi_3)\}_0 \\ & = g\lim_{r\to 0}\frac{e^{-\mu r}}{r^2}\left\{\mathbf{R}^2\left(\mu + \frac{1}{r}\right) - (\mathbf{R}\cdot\mathbf{r}_0)^2\left[\frac{1}{r} + (\mu r + 2)\left(\mu + \frac{1}{r}\right)\right]\right\} \end{aligned}$$

where $\mathbf{r}_0$ is a unit vector in the direction of $\mathbf{r}$. Before going to the

† These would also arise with the terms φ_4^2, ψ_4^2 etc., also because of the equation of motion (9).

†† This splitting-off of the static self energy could, of course, have been achieved also by the well-known transformation used by Møller and Rosenfeld, Dan. Mat. Fys. Medd. **17**, no. 8 (1940), which is more general than the procedure used here. With respect to the sign of the static self energy cf. the appendix.

limit $r \to 0$ we have† to average the product $(\mathbf{R} \cdot \mathbf{r}_0)^2$ over all directions of $\mathbf{r}_0$:

$$\overline{(\mathbf{R} \cdot \mathbf{r}_0)^2} = \tfrac{1}{3} \mathbf{R}^2.$$

If we further define the dynamic self energy††

$$\delta m = \lim_{r \to 0} \frac{1}{3} g^2 \left(\frac{1}{r} - \frac{3}{2} \mu \right), \tag{12}$$

we are left with

$$g\{\mathbf{R} \cdot \operatorname{grad} (\mathbf{R} \cdot \operatorname{grad} \varphi_3)\}_0 = - \mu^2 \mathbf{R}^2 \left(\delta m + \frac{1}{6} g^2 \mu \right).$$

If we now drop the index 1 of φ and ψ, we get as the starting point for the principal axis transformation the Hamiltonian

$$\begin{aligned} H = \frac{\mathbf{P}^2}{2m} + g\, (\mathbf{R} \cdot \operatorname{grad} \varphi)_0 - \frac{\mu^2}{2} \mathbf{R}^2 \left(\delta m + \frac{1}{6} g^2 \mu \right) \\ + \frac{1}{8\pi} \int \{(4\pi\psi)^2 + (\operatorname{grad} \varphi)^2 + \mu^2 \varphi^2\}\, d\tau. \end{aligned} \tag{13}$$

The principal axis transformation itself is then achieved in two steps [2]):

1. By a suitable linear canonical transformation of the 'coordinates' $\mathbf{R}, \varphi \to \mathbf{R}', \varphi'$ and the 'momenta' $\mathbf{P}, \psi \to \mathbf{P}', \psi'$ the particle and field quantities are decoupled.†††
The parameters of the transformation are determined by the postulate that the coefficients of the 'mixed' terms $\propto \mathbf{R}'\psi'$ and $\mathbf{P}'\psi'$ vanish.

2. Expanding the field quantities with the aid of a complete set of space functions *

$$\varphi' = \sum_n \varphi_n v_n(\mathbf{r}); \quad \psi' = \sum_n \psi_n u_n(\mathbf{r}), \tag{14}$$

$$\int u_n v_m d\tau = \delta_{mn}, \tag{15}$$

the Hamiltonian becomes simply a sum of uncoupled oscillators.

† One gets this prescription for taking the limit immediately if one starts with an extended charge distribution for the nucleon which finally goes over into a δ-function, cf. also e.g. ref. [3]).

†† The justification of this definition is given in the appendix.

††† Of course the new „particle" variables depend on the original particle *and* field variables, and the same holds for the new „field" variables.

* As it turns out, this set is non-orthogonal, i.e. $u_n \neq v_n$, cf. (31), (32).

Again the system of functions is determined by the postulate that the coefficients of the mixed terms $\varphi_n\varphi_m$, $\psi_n\psi_m$ $(m \neq n)$ vanish.

3. Decoupling of Field and Particle Variables

From the generating function

$$W(\mathbf{R},\mathbf{P}',\varphi',\psi) = \int w d\tau = a\mathbf{R}\cdot\mathbf{P}' + \int \mathbf{R}\cdot\mathbf{f}(\mathbf{r})\psi(\mathbf{r})\,d\tau + \int \mathbf{P}'\cdot\mathbf{h}(\mathbf{r})\varphi'(\mathbf{r})\,d\tau + \int \psi(\mathbf{r})\varphi'(\mathbf{r})\,d\tau,$$

where a is constant and $\mathbf{f}(\mathbf{r})$, $\mathbf{h}(\mathbf{r})$ are vector functions to be determined later, we get the canonical transformation

$$\varphi = -\frac{\partial w}{\partial \psi} = -\varphi' - \mathbf{R}\cdot\mathbf{f}$$

$$\mathbf{R}' = \frac{\partial W}{\partial \mathbf{P}'} = a\mathbf{R} + \int \mathbf{h}\varphi' d\tau$$

$$\psi' = -\frac{\partial w}{\partial \varphi'} = -\psi - \mathbf{P}'\cdot\mathbf{h}$$

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{R}} = a\mathbf{P}' + \int \mathbf{f}\psi d\tau,$$

or, resolved with respect to the original variables,

$$\begin{aligned}
\psi &= -\psi' - \mathbf{P}'\cdot\mathbf{h} \\
\mathbf{R} &= \frac{1}{a}\left(\mathbf{R}' - \int \mathbf{h}\varphi' d\tau\right) \\
\mathbf{P} &= a\mathbf{P}' - \int \mathbf{f}(\mathbf{h}\cdot\mathbf{P}')\,d\tau - \int \mathbf{f}\psi' d\tau \\
\varphi &= -\varphi' + \frac{1}{a}\mathbf{f}\cdot\int \mathbf{h}\varphi' d\tau - \frac{1}{a}\mathbf{R}'\cdot\mathbf{f}.
\end{aligned} \tag{16}$$

If particle and field variables are to be separated after this transformation, then $\psi' = \varphi' = 0$ must be a possible (classical) solution of the problem. In this case we get from (16)

$$\psi = -\mathbf{P}'\cdot\mathbf{h}, \quad \varphi = -\frac{1}{a}\mathbf{R}'\cdot\mathbf{f}.$$

Now

$$\psi = \frac{1}{4\pi}\dot{\varphi}, \quad \text{i.e.} \quad -\mathbf{P}'\cdot\mathbf{h} = -\frac{1}{4\pi a}\dot{\mathbf{R}}'\cdot\mathbf{f}.$$

If we choose the coefficient of P'^2 in the transformed Hamiltonian

to be $\frac{1}{2}$, then $\dot{\mathbf{R}}' = \mathbf{P}'$ and consequently $\mathbf{f} = 4\pi a\mathbf{h}$. So there remains

$$\begin{aligned}
\psi &= -\psi' - \mathbf{P}' \cdot \mathbf{h} \\
\mathbf{R} &= \frac{1}{a}\left(\mathbf{R}' - \int \mathbf{h}\varphi' d\tau\right) \\
\mathbf{P} &= a\mathbf{P}' - 4\pi a \int \mathbf{h}(\mathbf{h}\cdot\mathbf{P}')d\tau - 4\pi a \int \mathbf{h}\psi' d\tau \\
\varphi &= -\varphi' + 4\pi\mathbf{h}\cdot\int \mathbf{h}\varphi' d\tau - 4\mathbf{R}'\cdot\mathbf{h}.
\end{aligned} \tag{17}$$

Inserting this into the Hamiltonian (13) we get the following 'mixed' terms $\propto \mathbf{R}'\varphi'$:

$$\begin{aligned}
&-\frac{g}{a}\mathbf{R}'\cdot(\operatorname{grad}\varphi')_0 + \mu^2\int \mathbf{R}'\cdot\mathbf{h}\varphi' d\tau \\
&+\int \operatorname{grad}\varphi'\cdot\operatorname{grad}(\mathbf{R}'\cdot\mathbf{h})\,d\tau \\
&+\frac{4\pi g}{a}\mathbf{R}'\cdot\left(\operatorname{grad}\left(\mathbf{h}\cdot\int \mathbf{h}\varphi' d\tau\right)\right)_0 \\
&-4\pi\mu^2\int(\mathbf{R}'\cdot\mathbf{h})\left(\mathbf{h}\cdot\int\mathbf{h}\varphi' d\tau\right)d\tau \\
&-4\pi\int\operatorname{grad}\left(\mathbf{h}\cdot\int\mathbf{h}\varphi' d\tau\right)\cdot\operatorname{grad}(\mathbf{R}'\cdot\mathbf{h})d\tau \\
&+\frac{4\pi g}{a}(\operatorname{grad}(\mathbf{R}'\cdot\mathbf{h}))_0\cdot\int\mathbf{h}\varphi' d\tau \\
&+\frac{\mu^2\left(\delta m + \frac{1}{6}\mu g^2\right)}{a^2}\mathbf{R}'\cdot\int\mathbf{h}\varphi' d\tau.
\end{aligned} \tag{18}$$

It is seen immediately that the first three and the next three terms cancel each other if

$$\Delta\mathbf{R}'\cdot\mathbf{h} = \operatorname{div}\frac{g}{a}\delta(\mathbf{r})\mathbf{R}' + \mu^2\mathbf{R}'\cdot\mathbf{h}$$

i.e.

$$\Delta\mathbf{h} - \mu^2\mathbf{h} = \frac{g}{a}\operatorname{grad}\delta(\mathbf{r})$$

and therefore

$$\mathbf{h} = -\frac{g}{4\pi a}\operatorname{grad}\frac{e^{-\mu r}}{r}. \tag{19}$$

Furthermore

$$\begin{aligned}
\frac{4\pi g}{a}(\operatorname{grad}(\mathbf{R}'\cdot\mathbf{h}))_0 &= \frac{1}{a^2}g^2\mu^2\mathbf{R}'\cdot\frac{1}{3}\lim_{r\to 0}\left(\frac{1}{r} - \mu\right) \\
&= -\frac{\mu^2}{a^2}\left(\delta m + \frac{1}{6}\mu g^2\right)\mathbf{R}',
\end{aligned}$$

so that the last two terms of (18) vanish too.

The mixed terms $\propto \mathbf{P}'\psi'$ in the transformed Hamiltonian are

$$-\frac{4\pi a^2}{m}\left(\mathbf{P}' - 4\pi\int \mathbf{h}(\mathbf{h}\cdot\mathbf{P}')\,d\tau\right)\cdot\int \mathbf{h}\psi'\,d\tau$$

$$+ 4\pi\,\mathbf{P}'\int \mathbf{h}\psi' d\tau.$$

Because of

$$\int \mathbf{h}(\mathbf{h}\cdot\mathbf{P}')d\tau = \frac{1}{4\pi a^2}\delta m\mathbf{P}'$$

these terms vanish if

$$-\frac{a^2}{m}\left(1 - \frac{1}{a^2}\delta m\right) + 1 = 0.$$

From this we determine the value of a:

$$a^2 = m + \delta m = M, \tag{19a}$$

where $m + \delta m = M$ is the 'experimental' mass of the nucleon. Using all these results we get the transformed Hamiltonian

$$H = \tfrac{1}{2}\mathbf{P}'^2 + \frac{g^2}{2m}\left\{\int \frac{e^{-\mu r}}{r}\,\text{grad}\;\psi' d\tau\right\}^2$$
$$+ \frac{1}{8\pi}\int\{(4\pi\psi')^2 + (\text{grad}\;\varphi')^2 + \mu^2\varphi'^2\}d\tau. \tag{20}$$

4. Expansion of the Field

If we now make the transformation (14) which as a matter of fact is a canonical transformation too,[2]) we want to get the final result

$$H = \tfrac{1}{2}\mathbf{P}'^2 + \tfrac{1}{2}\sum_n (\psi_n^2 + \omega_n^2\varphi_n^2). \tag{21}$$

From this it follows that

$$\mathbf{R}' = \mathbf{P}' = \psi_m = \varphi_m = 0 \text{ for } m \neq n,$$

i.e. only ψ_n, $\varphi_n \neq 0$ must be a possible solution to our problem. In that case we get from (16) and (14)

$$\psi = -\psi_n u_n$$
$$\varphi = -\varphi_n\left(v_n - 4\pi\mathbf{h}\cdot\int \mathbf{h}v_n d\tau\right).$$

Now we get from (21)

$$\dot{\varphi}_n = \psi_n,$$

which gives, with the definition of ψ,

$$\dot{\varphi} = -\dot{\varphi}_n\left(v_n - 4\pi\mathbf{h}\cdot\int \mathbf{h}v_n d\tau\right)$$
$$= -\psi_n\left(v_n - 4\pi\mathbf{h}\cdot\int \mathbf{h}v_n d\tau\right)$$
$$= 4\pi\psi = -4\pi\psi_n u_n.$$

So we must have

$$4\pi u_n(\mathbf{r}) = v_n(\mathbf{r}) - 4\pi\mathbf{h}(\mathbf{r})\cdot\int \mathbf{h}v_n d\tau. \tag{22}$$

Now

$$4\pi\int \mathbf{h}u_n d\tau = \int \mathbf{h}v_n d\tau - 4\pi\left(\int \mathbf{h}v_n d\tau\cdot\int \mathbf{h}\right)\mathbf{h}d\tau$$
$$= \int \mathbf{h}v_n d\tau\left(1 - \frac{\delta m}{M}\right)$$
$$= \frac{m}{M}\int \mathbf{h}v_n d\tau.$$

This gives together with (22) and (19)

$$v_n = 4\pi u_n - \frac{g^2}{m}\operatorname{grad}\frac{e^{-\mu r}}{r}\cdot\int\frac{e^{-\mu r}}{r}\operatorname{grad} u_n d\tau. \tag{23}$$

The transformation (14) inserted into the Hamiltonian (20) gives the following coefficient for $\psi_n\ \psi_m$:

$$\frac{g^2}{2m}\int\frac{e^{-\mu r}}{r}\operatorname{grad}\ u_n d\tau\cdot\int\frac{e^{-\mu r}}{r}\operatorname{grad} u_m d\tau + 2\pi\int u_n u_m d\tau, \tag{24}$$

and for $\varphi_n\ \varphi_m$:

$$-\int v_n(\Delta v_m - \mu^2 v_m)\, d\tau. \tag{25}$$

These terms are intended to vanish for $n \neq m$. Eq. (24) is simply the orthogonality relation (15) because of (23). Then the terms (25) vanish too if

$$\Delta v_m - \mu^2 v_m = -\omega_m^2 u_m,$$

where ω_m is a constant which may depend on m. With (23) and the abbreviation

$$\mathbf{V}_n = \int\frac{e^{-\mu r}}{r}\operatorname{grad} u_n d\tau, \tag{26}$$

this gives

$$\Delta u_m + (\omega_m^2 - \mu^2)u_m = -\frac{g^2}{m}\mathbf{V}_m\cdot\operatorname{grad}\delta(\mathbf{r}). \tag{27}$$

If we put

$$u_m(\mathbf{r}) = \operatorname{div} \mathbf{V}_m w_m(\mathbf{r}) = \mathbf{V}_m \cdot \operatorname{grad} w_m(\mathbf{r}) \tag{28}$$

we get

$$\Delta w_m + (\omega_m^2 - \mu^2) w_m = -\frac{g^2}{m}\delta(\mathbf{r}). \tag{29}$$

From (26) and (28) it follows with (27) or (29), according to the two different possibilities in performing the partial integration,

$$\frac{1}{3}\int \frac{e^{-\mu r}}{r}\Delta w_m \, d\tau$$

$$= -\frac{1}{m}\left(\delta m + \frac{1}{6}\mu g^2\right) - \frac{1}{3}(\omega_m^2 - \mu^2)\int \frac{e^{-\mu r}}{r} w_m \, d\tau = 1$$

or

$$= \frac{1}{3}\int w_m \Delta \frac{e^{-\mu r}}{r} d\tau$$

$$= -\frac{4\pi}{3} w_m(0) + \frac{\mu^2}{3}\int \frac{e^{-\mu r}}{r} w_m \, d\tau = 1.$$

So from these two equations we can eliminate the integral and finally get

$$w_m(0) = -\frac{3}{4\pi}\left(1 + \frac{\mu^2}{\omega_m^2 - \mu^2} \cdot \frac{M + \frac{1}{6}\mu g^2}{m}\right). \tag{30}$$

The solution of (29) must have the form

$$w_m = A_m \frac{\sin(k_m r - \eta_m)}{r} = \frac{A_m}{r}(\sin k_m r \cos \eta_m - \cos k_m r \sin \eta_m) \tag{31}$$

where

$$k_m^2 = \omega_m^2 - \mu^2$$

and the boundary condition (2) requires

$$\sin(k_m L - \eta_m) = 0$$

or

$$k_m L - \eta_m = (m + \tfrac{1}{2})\pi, \quad m \text{ integer}. \tag{32}$$

Because of (29) we must have

$$A_m = -\frac{g^2}{m}\frac{1}{4\pi \sin \eta_m}, \tag{33}$$

so that from (30) and (31) we get

$$w(0) = -\frac{g^2}{m}\frac{1}{4\pi \sin \eta_m}\left\{k_m \cos \eta_m - \left(\frac{3\delta m}{g^2} + \frac{3}{2}\mu\right)\sin \eta_m\right\} \tag{34}$$
$$= -\frac{3}{4\pi}\left(1 + \frac{\mu^2}{k_m^2}\frac{M + \frac{1}{6}\mu g^2}{m}\right)$$

which gives together with (32)†

$$\cot \eta_m = -\tan k_m L = \frac{\mu}{k_m} + \frac{\mu^2 + k_m^2}{k_m^3}\left(\frac{3M}{g^2} + \tfrac{1}{2}\mu\right). \tag{35}$$

The normalization of the functions u_n, v_n gives for large L by a straightforward calculation ††

$$\int u_n v_n d\tau = \mathbf{V}_n^2 \left(\frac{g^2}{m}\right)^2 \frac{k_n^2}{6}\frac{L}{\sin^2 \eta_n} = 1, \tag{36}$$

so that with (33) we have

$$A_n \mathbf{V}_n^{(\lambda)} = \frac{\sqrt{6}}{4\pi k_n}\frac{1}{\sqrt{L}}\mathbf{e}_\lambda, \quad \lambda = 1, 2, 3,$$

where the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form an orthonormal basic system in space. So we finally get †††

$$u_n^{(\lambda)} = \frac{\sqrt{6}}{4\pi k_n}\frac{1}{\sqrt{L}}\mathbf{e}_\lambda \cdot \operatorname{grad}\frac{\sin(k_n r - \eta_n)}{r} \tag{37}$$

$$v_n^{(\lambda)} = \frac{\sqrt{6}}{4\pi k_n}\frac{1}{\sqrt{L}}\mathbf{e}_\lambda \cdot \operatorname{grad}\frac{\sin(k_n r - \eta_n) + e^{-\mu r}\sin \eta_n}{r} \tag{38}$$

$$k_n L - \eta_n = (n + \tfrac{1}{2})\pi$$

$$\int u_u^{(\lambda)} v_m^{(\mu)} d\tau = \delta_{mn}\delta_{\lambda\mu}$$

$$H = \tfrac{1}{2}\mathbf{P}'^2 + \sum_n \sum_{\lambda=1}^{3}\{(\psi_n^{(\lambda)})^2 + \omega_n^2(\varphi_n^{(\lambda)})^2\} \tag{39}$$

$$k_n^2 = \omega_n^2 - \mu^2 \tag{40}$$

† This important formula was derived here in a manner different from ref. [2]) and which seems to be slightly more convenient. It is easily shown that (35) is also necessary for fulfilling the orthogonality relation (15).

†† This does not hold for imaginary k, where we get (42) (43) instead of (37) (38).

††† Originally the expansion (14) is an expansion with respect to spherical harmonics. But, as we see, only the p-waves are coupled to the nucleon in this approximation, so that all the other terms have been left out from the Hamiltonian (39). This corresponds to the electromagnetic case, cf. ref.[1]) or ref.[2]).

$$\cot \eta_n = -\tan k_n L = \frac{\mu}{k_n} + \frac{\mu^2 + k_n^2}{k_n^3}\left\{\frac{3M}{g^2} + \tfrac{1}{2}\mu'\right\}. \qquad (41)$$

We note that eq. (41) for the determination of the k_n is independent of the sign of k_n, i.e. if k_n is a solution, then $-k_n$ is a solution too. As this change of the sign of k brings no change of the functions (37), (38), we can restrict ourselves to positive values for k.

5. Discussion

Every normal mode of vibration following from the Hamiltonian (39) represents a standing meson wave and an oscillartory motion of the nucleon, i.e. a scattering state, with the following exceptions:

1) H allows

$$\dot{\mathbf{P}}' = 0, \qquad \dot{\mathbf{R}}' = \mathbf{P}' = \text{const.}$$

This gives with (16)

$$\dot{\mathbf{R}} = \frac{1}{M}\dot{\mathbf{R}}' = \text{const}$$

$$\varphi = \frac{g}{a}\mathbf{R}' \cdot \operatorname{grad}\frac{e^{-\mu r}}{r} = g\mathbf{R} \cdot \operatorname{grad}\frac{e^{-\mu r}}{r}.$$

Within the limits of our approximation this represents simply a nucleon moving with constant velocity $\dot{\mathbf{R}}$ and surrounded by the static field.

2) The equation (41) for the wave numbers k has also an imaginary solution. If we put

$$k = i\varkappa, \qquad \left\{\frac{3M}{\mu g^2} + \tfrac{1}{2}\right\} = \alpha,$$

$$\sqrt{1 - \frac{\omega^2}{\mu^2}} = \frac{\varkappa}{\mu} = p,$$

we get for large L ($\tanh \varkappa L = 1$)†

$$p^3 - (1 + \alpha)\, p^2 + \alpha = (p - 1)\,(p^2 - \alpha\, p - \alpha) = 0.$$

Now $p = 1$, i.e. $\varkappa = \mu$, is not an admissible solution as we have, for imaginary $k = i\varkappa$,

$$u = \frac{\sqrt{6}}{4\pi}\sqrt{\frac{\mu + \varkappa}{2\mu^2 - \varkappa(\varkappa + \mu)}}\,\mathbf{e} \cdot \operatorname{grad}\frac{e^{-\varkappa r}}{r} \qquad (42)$$

† The argument referring to the sign of k (cf. (41)) holds for $\varkappa$ too, so that we restrict ourselves to positive values of $\varkappa$.

$$v = \sqrt{6}\sqrt{\frac{\mu+\varkappa}{2\mu^2-\varkappa(\varkappa+\mu)}}\,\mathbf{e}\cdot\operatorname{grad}\frac{e^{-\varkappa r}-e^{-\mu r}}{r} \tag{43}$$

and v vanishes for $\varkappa = \mu$. There then remains

$$p^2 - \alpha p - \alpha = 0$$

with the solution †

$$p = \frac{\alpha}{2} + \sqrt{\alpha + \frac{\alpha^2}{4}}. \tag{44}$$

There are two essentially different possibilities:

a) $p > 1$. This means that k *and* ω are imaginary and therefore leads to the so-called 'runaway solution', well known from the corresponding case of an electron coupled to the electromagnetic field (cf e.g. ref. [3]).

b) $p < 1$. This means that k is imaginary, but ω is real. Therefore the field is falling off exponentially in space (cf. (42), (43)), but the time dependence of the field and the nucleon coordinate is, in contrast to the runaway solution, simply oscillatory. A quantization of the Hamiltonian (39) which can be carried through immediately gives accordingly an infinite series of excited states of the nucleon with level spacing $\hbar\omega$.

From (44) it follows that

$$\begin{aligned} p &\approx \sqrt{\alpha} && \text{if } \alpha \ll 1, \\ p &\approx \alpha && \text{if } \alpha \gg 1, \\ p &= 1 && \text{for } \alpha = \tfrac{1}{2}. \end{aligned}$$

But from the definition of α we see that always $\alpha > \frac{1}{2}$, and therefore $p > 1$. This means according to our argumentation above that in our problem we only get the runaway solutions, but no excited states.

It is to be noted that the formulae (37) — (41) no longer depend on m or δm, but only on the experimental mass M, which means that the procedure (19a), at first sight rather formal, is indeed the usual mass renormalization, but of course in a very simple non-relativistic form. From the phase shift η (cf. (35)) we get immediately the cross section σ for the scattering of scalar mesons by free nucleons in nonrelativistic approximation:

† The solution $p = (\alpha/2) - \sqrt{\alpha + (\alpha/2)^2}$ is meaningless because we restricted ourselves to positive values of $\varkappa$.

$$\sigma = \frac{4\pi}{k^2}(2l+1)\sin^2\eta = 12\pi \frac{1}{k^2 + \left(\mu + \gamma + \frac{\gamma\mu^2}{k^2}\right)^2},$$

where $\gamma = \frac{3M}{g^2} + \frac{1}{2}\mu$ and $l = 1$ (p-waves).

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Appendix

DEFINITION OF THE DYNAMIC SELF ENERGY.

The *relativistic* Hamiltonian for our problem is

$$H = (m + g\varphi)\sqrt{1 + \frac{\mathbf{P}^2}{(m+g\varphi)^2}} + \frac{1}{8\pi}\int \{(4\pi\psi)^2 + (\operatorname{grad}\varphi)^2 + \mu^2\varphi^2\}\,d\tau \tag{A1}$$

and from this we get in non-relativistic approximation

$$H \approx m + g\varphi + \tfrac{1}{2}\frac{\mathbf{P}^2}{(m+g\varphi)} + \frac{1}{8\pi}\int \{(4\pi\psi)^2 + (\operatorname{grad}\varphi)^2 + \mu^2\varphi^2\}d\tau.$$

If now we make the Taylor-expansion (3) for φ and only retain the quadratic terms, we have to take into account that φ_3 (0) (cf. (7) ff.) is an (infinite) *constant*, independent of all dynamic variables describing our system, so that we have to write

$$H \approx m + g\varphi + \tfrac{1}{2}\frac{\mathbf{P}^2}{m'} + \frac{1}{8\pi}\int \{(4\pi\psi)^2 + (\operatorname{grad}\varphi)^2 + \mu^2\varphi^2\}d\tau \tag{A2}$$

with $m' = m + g\,\varphi_3(0)$.

This shows that in deriving the Hamiltonian (1) we have (not

quite correctly) left out the first term (m), and further m' is simply denoted by m. This, of course, does not alter the dynamical aspects of our problem, but for the total static energy we now get from (A2)

$$H_{\text{stat}} = m' + \frac{g}{2} \lim_{r \to 0} \left(\frac{1}{r} - \mu \right). \tag{A3}$$

Here the second term on the right hand side would represent the proper static self energy and is, of course, positive. If we denote the energy momentum tensor of the meson field by T_μ^ν,† then the total four-momentum P_μ of the field is generally given by

$$P_\mu = \int T_\mu^4 dV,$$

where dV is the three dimensional volume element, and we have to integrate for $t = \text{const}$. Let us then denote the energy momentum tensor of the static meson field enclosing a nucleon at rest, taken in the rest system, by $T_\mu'^\nu$, and the same quantity, taken in a system of reference moving with the velocity β in the x_1 direction, by $T_\mu''^\nu$; we have according to well-known transformation laws

$$P_1'' = \int T_1''^4 dV'' = \frac{\beta}{\sqrt{1-\beta^2}} \int (T_4'^4 - T_1'^1) dV'$$

if we take into account that $\int T_1'^4 dV' = 0$.

So we have non-relativistically

$$P_1'' = \beta \int (T_4'^4 - T_1'^1) dV'$$

and therefore get the following definition for the total dynamic mass††

$$\begin{aligned} M &= m + \int (T_4'^4 - T_1'^1) dV' \\ &= m - g^2 \lim_{r \to 0} \left\{ \frac{2}{3} \frac{1}{r} - \frac{\mu}{2} \right\} \\ &= m' + \frac{g^2}{3} \lim_{r \to 0} \left\{ \frac{1}{r} - \frac{3}{2} \mu \right\}, \end{aligned}$$

which justifies (12) and (19a).

† We use here the notation $a_1 = a^1$ etc., but $a_4 = -a^4$.

†† We have after a straightforward calculation (cf e.g. ref. [6])

$$\int T_4'^4 dV' = \frac{g^2}{2} \lim_{r \to 0} \left(\mu - \frac{1}{r} \right)$$

$$\int T_1'^1 dV' = \frac{g^2}{6} \lim_{r \to 0} \frac{1}{r};$$

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THEORY OF THE IONIZATION CASCADE

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Abstract: The statistical problem connected with the *ionization cascade*, that is, the cascade of knock-on electrons produced by a fast primary particle passing through matter, is to obtain the distribution of quantities of interest connected with the cascade, such as the number of electrons in a given energy range, given the ionization cross-sections. An exact solution of this problem is given in very general terms, which will yield all such distributions. With the help of suitable approximations, the distributions of (1) the number of fast knock-on electrons of energy $\geq$ given value W, and (2) the total number of ion-pairs produced by the primary particle, are obtained in a form suitable for numerical computation.

1. Introduction

A fast primary ionizing particle passing through an absorber ejects knock-on electrons which, if energetic enough, will ionize further atoms, thus giving rise to the typical 'delta rays' which branch off the track of such a particle in a cloud-chamber or nuclear emulsion. In a previous paper [1]) (referred to henceforth as I), the distribution of the total number of ion-pairs produced by such a particle was obtained. This was based on an approximate expression for the number distribution of ion-pairs produced at each ionizing collision. The purpose of the present paper is to give a general theory of the *number distributions of electrons of given energies* produced by the primary ionizing particle.

The theory proceeds in two steps:

1. An iteration method is given for calculating the number distributions of the *knock-on cascade*: i.e. of the electrons of given energies produced by a primary knock-on of energy E; this incidentally permits an improved derivation of the distribution of the total number of ion-pairs produced at each collision which entered into the calculations of I.

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2. The distributions obtained in the first step are compounded, using the probability per unit time that the ionizing primary ejects a knock-on of energy E, to yield the final number distributions of the *ionization cascade*: i.e. of the electrons of given energies produced by the primary ionizing particle.

2. The Knock-on Electron Cascade

The production of electrons by a single primary knock-on constitutes a multiplicative cascade process analogous to the nucleon cascade [2]): a fast primary knock-on will ionize an atom, producing thereby secondary electrons, which may in turn eject further electrons, and so on, the process terminating when all its end products are either too slow to ionize further, or else have escaped or have been absorbed. We shall assume as in I that

a) the absorber is homogeneous;

b) successive ionizing collisions are statistically independent;

c) an electron will ionize only if its energy is greater than a mean ionization potential I characteristic of the absorber atoms, and will contribute thereby two electrons to the cascade, escape and absorption being neglected;

d) we may neglect radiation loss and other extraneous effects, such as the increase in ionization due to the acceleration of electrons in electrode fields, and photoelectric effects in the absorber or in the walls of its container.

These assumptions are suitable for applications of the theory to ionization or cloud-chambers.

It is possible to obtain the solution of the knock-on electron cascade problem in very general terms by introducing the *characteristic functional* of its distribution†, defined as follows: let $n(W, E, t)$ be the number of electrons of energy $\geqq W$ created by a primary knock-on of energy E in time t; let $\theta(W)$ be an arbitrary real function of W; the characteristic functional of the distribution is

$$\Phi[\theta(W), E, t] = \mathscr{E}\left\{\exp\left[i\int_0^E \theta(W)dn(W, E, t)\right]\right\}, \qquad (2.1)$$

where $\mathscr{E}$ means expectation value. If $q(W_1, W_2, \ldots, W_n, E, t)\, dW_1 \ldots dW_n$ is the probability that the primary produces exactly n electrons with energies in the ranges $(W_1, W_1 + dW_1), \ldots$

† See e.g. Bartlett [10]) for a review of the applications of characteristic functionals to such problems.

$(W_n, W_n + dW_n)$, with the W_i ordered so that $W_1 \leqq W_2 \leqq \ldots \leqq W_n$ in order to take care of the indistinguishability of the electrons [2, 3]), then

$$\Phi[\theta(W), E, t] = \sum_{n=1}^{\infty} \int_0^E dW_n \int_0^{W_n} dW_{n-1} \cdots \int_0^{W_2} dW_1 e^{i \sum_{k=1}^{n} \theta(W_k)} q(W_1, \ldots, W_n, E, t). \quad (2.2)$$

From Φ we can obtain all distributions of interest connected with the cascade: thus, the probability generating function for the number of electrons of energy $\geqq W$ is

$$G(z, W, E, t) = \sum_{n=0}^{\infty} z^n q_n(W, E, t) = \Phi[\theta_z(U), E, t], \quad (2.3)$$

where $q_n(W,E,t,)$ is the probability that a primary knock-on of energy E produce in t n electrons of energies $\geqq W$,

$$\theta_z(U) = \begin{cases} \frac{1}{i} \log z & \text{when } U \geqq W, \\ 0 & \text{when } U < W. \end{cases} \quad (2.4)$$

The moments of this distribution are given by

$$\mu_n(W, E, t) = \overline{n^k}(W, E, t) = \left[\left(-\frac{\partial}{\partial s}\right)^k M(s, W, E, t)\right]_{s=0}, \quad (2.5)$$

where

$$M(s, W, E, t) = G(e^{-s}, W, E, t) = \Phi[\theta_s(U), E, t], \text{ with } i\,\theta_s(U) = \begin{cases} -s \text{ if } U \geqq W, \\ 0 \text{ if } U < W. \end{cases} \quad (2.6)$$

The Laplace inverse of the total energy distribution $\varrho(U, E, t)$ of the cascade is given by

$$M(\alpha, E, t) = \int_0^{\infty} e^{-\alpha U} \varrho(U, E, t)\, dU = \Phi[\theta_\alpha(U), E, t], \quad (2.7)$$

with $i\theta_\alpha(U) = -\alpha U$; and so on. In what follows, we shall use the abbreviated notation $\Phi[E, t]$ for the characteristic functional whenever its dependence on the auxiliary function $\theta(U)$ does not come into play.

The characteristic functional Φ satisfies a 'first collision' or 'G-equation' (analogous to Jánossy's G-equation for the nucleon cascade [9]; see e.g. Bartlett [10])) which is obtained by considering the variation of Φ in the initial time interval $(0,\delta t)$. Let $\sigma(U, E)dU$ be

the differential cross-section for an electron of energy E to collide with an atom of the absorber and lose energy between U and $U + dU$. For large U, the classical (Thomson) cross-section is a good approximation:

$$\sigma(U, E)dU = \frac{2\pi\varepsilon^4 Z}{\mu v^2}\frac{dU}{U^2}, \tag{2.8}$$

where Z is the atomic number of the absorber atoms, μ the mass, v the velocity of the electron, ε its charge. For U of the order of or less than I, account must be taken of quantal resonance effects arising from the binding of the orbital electrons (this will be discussed later on). We must distinguish between *ionizing* and *non-ionizing* collisions. We write for their rates respectively

$$\begin{gathered}\alpha_i(E) = Nv\int_I^E \sigma(U, E)dU,\ \alpha_e(E) = Nv\int_0^I \sigma(U, E)dU,\\ \alpha(E) = \alpha_i(E) + \alpha_e(E),\end{gathered} \tag{2.9}$$

where N is the number density of absorber atoms. The probability densities for the energy loss U *given* that an ionizing or a non-ionizing collision has occurred are respectively

$$\begin{aligned}\varphi_i(U, E) &= \begin{cases} Nv\sigma(U, E)/\alpha_i(E) & \text{if } U \geqq I,\\ 0 & \text{if } U < I,\end{cases}\\ \varphi_e(U,E) &= \begin{cases} Nv\sigma(U, E)/\alpha_e(E) & \text{if } U < I,\\ 0 & \text{if } U \geqq I,\end{cases}\\ \alpha(E)\varphi(U,E) &= \begin{cases} \alpha_i(E)\varphi_i(U, E) & \text{if } U \geqq I,\\ \alpha_e(E)\varphi_e(U, E) & \text{if } U < I.\end{cases}\end{aligned} \tag{2.10}$$

In the initial time interval $(0, \delta t)$, there is a probability $\{1-\alpha(E)\}\,\delta t$ that the primary electron will suffer no collision, a probability $\alpha_e(E)\delta t\,\varphi_e(U, E)dU$ that it will lose energy between U and $U + dU$ in a *non-ionizing* collision, and a probability $\alpha_i(E)\delta t\,\varphi_i(U, E)dU$ that it will lose energy between U and $U + dU$ in an *ionizing* collision, giving rise thereby to two cascade electrons of energies respectively $E - U$ and $U - I$, which will act as the primaries of two independent cascades; hence

$$\begin{aligned}\Phi[E, t] &= [1 - \alpha(E)\delta t]\Phi[E, t - \delta t]\\ &\quad + \alpha_e(E)\delta t\int_0^I \Phi[E - U, t - \delta t]\varphi_e(U, E)dU\\ &\quad + \alpha_i(E)\delta t\int_0^E \Phi[E-U, t-\delta t]\Phi[U-I, t-\delta t]\varphi_i(U, E)dU.\end{aligned} \tag{2.11}$$

It follows that

$$\frac{\partial}{\partial t}\Phi[E,t]=\alpha_s(E)\left\{-\Phi[E,t]+\int_0^I \Phi[E-U,t]\varphi_s(U,E)dU\right\}$$
$$+\alpha_i(E)\left\{-\Phi[E,t]+\int_I^E \Phi[E-U,t]\Phi[U-I,t]\varphi_i(U,E)dU\right\}. \tag{2.12}$$

An approximate solution of (2.12) is obtained by setting $\alpha_s(E)=0$, i.e. neglecting energy dissipation by non-ionizing collisions. This is certainly justified if we are interested only in the high energy part of the cascade. We now remark that if $\Phi[E,t]$ is known for $E<kI$, where k is an integer, then the integral in the second curly bracket in the right-hand-side of (2.12) is a known function of E, say $F[E,t]$, in the range $kI\leqq E<(k+1)I$; hence in this range of E, Φ satisfies the equation

$$\left[\frac{\partial}{\partial t}+\alpha_i(E)\right]\Phi[E,t]=\alpha_i(E)F[E,t]$$
$$=\alpha_i(E)\int_I^E \Phi[E-U,t]\Phi[U-I,t]\varphi_i(U,E)dU, \tag{2.13}$$

whose solution with the initial condition

$$\Phi[E,0]=e^{i\theta(E)} \tag{2.14}$$

(expressing the fact that at $t=0$, the cascade consists of just the initial electron of energy E) is

$$\Phi[E,t]=e^{i\theta(E)-\alpha_i(E)t}+\alpha_i(E)\int_0^t e^{-\alpha_i(E)(t-\tau)}F(E,\tau)d\tau. \tag{2.15}$$

Since

$$\Phi[E,t]=e^{i\theta(E)} \text{ for } E<I, \tag{2.16}$$

because there can be no ionizing collisions in this range, we can use (2.15) as an iteration relation to obtain $\Phi[E,t]$ in the successive ranges $(kI,(k+1)I)$, $k=1,2,\ldots$. Thus, for $I\leqq E<2I$,

$$\Phi[E,t]=e^{i\theta(E)-\alpha_i(E)t}+\alpha_i(E)\int_I^E e^{i[\theta(E-U)+\theta(U-I)]}\varphi_i(U,E)dU. \tag{2.17}$$

The solution of (2.12) without neglection of energy dissipation by non-ionizing collisions can be constructed in the following way. Let $\chi(W,E,t)\,dW$ be the probability that the primary electron of initial energy E suffer *no ionizing collisions*, but make a transition to an energy in the range W, $W+dW$ *by non-ionizing collisions* in the time t; χ satisfies the equation

$$\frac{\partial}{\partial \tau}\chi(W,E,t)=-\alpha(E)\chi(W,E,t)+\alpha_e(E)\int_0^I \chi(W,E-U,t)\varphi_e(U,E)dU, \tag{2.18}$$

whose solution is [2])

$$\chi(W,E,t)=\sum_{n=0}^{\infty}\chi_n(W,E,t), \tag{2.19}$$

where

$$\chi_0(W,E,t)=e^{-\alpha(E)t}\delta(W-E), \tag{2.20}$$

$$\chi_n(W,E,t)=$$

$$\int_0^t d\tau\int_0^I dU\chi_{n-1}(W,E-U,t-\tau)\,e^{-\alpha(E)\tau}\alpha_e(E)\varphi_e(U,E),\ (n\geqq 1). \tag{2.21}$$

We can now write that

$$\Phi[E,t]=\int_0^E e^{t\theta(W)}\chi(W,E,t)dW$$

$$+\int_0^t d\tau\int_I^E dW\int_I^W dU\Phi[W-U,t-\tau]\Phi[U-I,t-\tau]\Big| \tag{2.22}$$

$$\varphi_i(U,W)\alpha_i(W)\chi(W,E,\tau).$$

The first term in the right-hand-side of (2.22) is the contribution to Φ which arises if *no* ionizing collisions occur in t. The probability that the first ionizing collisions occur between τ and $\tau+d\tau$, and give rise to two electrons of energies respectively in the ranges $(U-I,\ U+dU-I)$ and $(W-U,\ W-U-dU)$ is

$$\varphi_i(U,W)\alpha_i(W)\chi(W,E,\tau)dWdUd\tau.$$

Multiplying this by the characteristic functionals of the two resultant cascades, and integrating over all intermediate times τ and energies W,U, we obtain the second term, i.e. the contribution to Φ from the *first* ionizing collision in t. One easily verifies by substitution that (2.22) satisfies (2.12).

We now remark that (2.22) can be used in exactly the same manner as our previous equation (2.15) to calculate Φ by iteration for increasing values of E, starting the iteration with

$$\Phi[E,t]=\int_0^E e^{t\theta(W)}\chi(W,E,t)dW \text{ for } E<I. \tag{2.23}$$

This therefore completes the exact solution of (2.12).

If the mean number of non-ionizing collisions in t is large (and this will usually be the case in practice for time intervals of interest)

the solution (2.19) converges very slowly. On the other hand, we know that in this case χ is approximately Gaussian, because the total energy loss $E - W$ in t results from a large number of *small* losses ($< I$), and hence that a good approximation to equation (2.18) is

$$\frac{\partial}{\partial t}\chi(W,E,t)=\left\{-\alpha_i(E)-a(E)\frac{\partial}{\partial E}+\tfrac{1}{2}b(E)\frac{\partial^2}{\partial E^2}\right\}\chi(W,E,t), \quad (2.24)$$

where

$$a(E)=\alpha_e(E)\int_0^I U\varphi_e(U,E)dU, \quad (2.25)$$

$$b(E)=\alpha_e(E)\int_0^I U^2\varphi_e(U,E)dU. \quad (2.26)$$

This so-called 'diffusion approximation' is obtained by substituting the first three terms of the Taylor expansion

$$\chi(W,E-U,t)=\chi(W,E,t)-U\frac{\partial}{\partial E}\chi(W,E,t)+\tfrac{1}{2}U^2\frac{\partial^2}{\partial E^2}\chi(W,E,t)-\ldots$$

in the integrand in the right-hand-side of (2.18). This approximation has the added advantage compared to the original equation (2.18) that whereas φ_e is a complicated function, strongly dependent on the atomic properties of the absorber, the coefficients $a(E)$ and $b(E)$ have well-known and comparatively simple approximate expressions [5, 6, 7])

$$a(E)\simeq 2\pi NZ\frac{\varepsilon^4}{\mu v}\left[\log\frac{2\mu v^2}{(1-v^2/c^2)I}-\frac{v^2}{c^2}\right], \quad (2.27)$$

$$b(E)\simeq 2\pi NZ\frac{\varepsilon^4}{\mu v}I. \quad (2.28)$$

The diffusion approximation can also be introduced directly into our original equation (2.12), which then becomes

$$\frac{\partial}{\partial t}\Phi[E,t]=\left\{-a(E)\frac{\partial}{\partial E}+\tfrac{1}{2}b(E)\frac{\partial^2}{\partial E^2}\right\}\Phi[E,t]$$

$$+\alpha_i(E)\left\{-\Phi[E,t]+\int_I^E\Phi[E-U,t]\Phi[U-I,t]\varphi_i(U,E)dU\right\}. \quad (2.29)$$

Set $a=b=0$ and we are back at our first approximations for Φ; $\chi(W,E,t)=\delta(W-E)$ in this case, and (2.22) reduces to (2.15). A second approximation is obtained by setting only $b=0$, i.e.

neglecting the *fluctuations* of energy loss by non-ionizing collisions. The solution of (2.24) with $b = 0$ is

$$\chi(W, E, t) = \delta[W - \mathscr{E}(t, E)] \exp\left\{ - \int_{\mathscr{E}(t,E)}^{E} \frac{\alpha_i(U)dU}{a(U)} \right\}, \quad (2.30)$$

where $\mathscr{E}(t,E)$, defined implicitly by the relation

$$t(\mathscr{E}, E) = \int_{\mathscr{E}}^{E} \frac{dU}{a(U)}, \quad (2.31)$$

is the energy to which the initial energy E is degraded in time t by non-ionizing collisions. The first factor in the right-hand-side of (2.30) expresses the fact that given that an electron of initial energy E suffers no ionizing collision in t, there is a probability unity that its energy will be $W = \mathscr{E}(t, E)$; the second factor is the probability that no ionizing collisions occur in t. Substituting (2.30) in (2.22) we find that

$$\begin{aligned} \Phi[E, t] &= \exp\left\{ i\theta(\mathscr{E}(t)) - \int_{\mathscr{E}(t)}^{E} \frac{\alpha_i(U)dU}{a(U)} \right\} \\ &+ \int_0^t dt \int_I^{\mathscr{E}(t)} dU \Phi[\mathscr{E}(\tau) - U, t - \tau]\Phi[U - I, t - \tau] \quad (2.32) \\ &\alpha_i(\mathscr{E}(\tau))\varphi_i(U, \mathscr{E}(\tau)) \exp\left\{ - \int_{\mathscr{E}(\tau)}^{E} \frac{\alpha_i(V)dV}{a(V)} \right\}, \end{aligned}$$

where $\mathscr{E}(t)$ stands throughout for $\mathscr{E}(t, E)$.

An alternative form of this result is obtained by solving (2.29) with $b = 0$:

$$\begin{aligned} \Phi[E, t] &= \exp\left\{ i\theta(\mathscr{E}(t)) - \int_{\mathscr{E}(t)}^{E} \frac{\alpha_i(U)dU}{a(U)} \right\} \\ &+ \int_I^E F(U, t) e^{-\int_U^E \frac{\alpha_i(W)dW}{a(W)}} \frac{\alpha_i(U)}{a(U)} dU \quad (2.33) \\ &- \int_I^{\mathscr{E}(t)} F(U, 0) e^{-\int_U^E \frac{\alpha_i(V)dV}{a(V)}} \frac{\alpha_i(U)}{a(U)} dU, \end{aligned}$$

where $F(E,t)$ is defined by (2.13).

Equation (2.24) will have to be solved numerically if we wish to take an exact account of the non-ionizing energy loss fluctuations. However, an approximate solution of the equivalent adjoint

equation

$$\frac{\partial}{\partial t}\chi(W,E,t)=\left\{-\alpha_i(W)+\frac{\partial}{\partial W}a(W)+\tfrac{1}{2}\frac{\partial^2}{\partial W^2}b(W)\right\}\chi(W,E,t) \quad (2.34)$$

is obtained by substituting $\mathscr{E}(t,E)$ defined by (2.31) for W in $\alpha_i(W)$, $a(W)$ and $b(W)$, namely†

$$\chi(W,E,t)=[2\pi\sigma^2(\tau,E)]^{-\frac{1}{2}}\exp\left\{-\frac{[E-W-m(t,E)]^2}{2\sigma^2(t,E)}-\lambda(t,E)\right\} \quad (2.35)$$

This is a Gaussian distribution for the energy loss $E-W$ with mean and standard deviation

$$m(t,E)=\int_0^t a[\mathscr{E}(\tau,E)]d\tau,\ \sigma(t,E)=\left[\int_0^t b[\mathscr{E}(\tau,E)]d\tau\right]^{\frac{1}{2}}, \quad (2.36)$$

multiplied by the probability $\exp[-\lambda(t,E)]$ that no ionizing collision occur in t, where

$$\lambda(t,E)=\int_0^t \alpha_i[\mathscr{E}(\tau,E)]d\tau. \quad (2.37)$$

3. The Ionization Electron Cascade

A fast primary ionizing particle which is not itself an electron acts as a *line source* of primary knock-on electrons, each of which initiates an independent electron cascade of the type described in the last section[8]). We add to the assumptions of section 2 that

a) the successive ionizing collisions of the primary particle are independent;

b) the loss of energy of the primary particle by all causes (including ionization) is negligible compared to its initial energy in a time interval t of interest. We write $\Psi[\theta(U),t]$ for the characteristic functional of the electron cascade initiated by the primary particle.

Let now $\sigma_p(U)dU$ be the differential cross-section for the primary to collide with an absorber atom and lose energy between U and $U+dU$. For U large, and smaller than the maximum transferable energy, this has the same form as (2.8), but with v now denoting the velocity of the primary particle, which we assume remains constant in t. Let

$$\alpha_p=Nv\int_I^\infty \sigma_p(U)dU;\ \varphi_p(U)=\begin{cases}Nv\sigma_p(U)/\alpha_p & \text{for } U\geqq I,\\ 0 & \text{for } U<I,\end{cases} \quad (3.1)$$

† This is the well-known point-source solution for the diffusion equation with a convection term and time-dependent coefficients.

be respectively the ionization rate (assumed constant) and probability density per ionizing collision for an energy loss U, N being as before the number density of absorber atoms.

We consider now the variation of $\Psi[t]$ in the initial time interval $(0, \delta t)$. There is a probability $1 - \alpha_p \delta t$ that the primary particle suffers no ionizing collision in this interval, and a probability $\alpha_p \delta t \varphi_p(U) dU$ that it produces a knock-on of energy between $U - I$ and $U - I + dU$, which initiates an independent electron cascade whose characteristic functional at time t is $\Phi[U - I, t]$. Hence

$$\Psi[t] = (1 - \alpha_p \delta t)\Psi[t - \delta t] + \alpha_p \delta t \Psi[t - \delta t] \int_I^\infty \Phi[U - I, t - \delta t] \varphi_p(U) dU. \quad (3.2)$$

It follows that $\Psi[t]$ satisfies the first-collision equation

$$\frac{\partial}{\partial t} \Psi[t] = \alpha_p \{\Phi_p[t] - 1\} \Psi[t], \quad (3.3)$$

where

$$\Phi_p[t] = \int_I^\infty \Phi[U - I, t] \varphi_p(U) dU \quad (3.4)$$

is the characteristic functional of the knock-on cascade produced in a single ionizing collision of the primary particle. The solution of (3.3) with the initial condition $\Psi[0] = 1$ is

$$\Psi[t] = \exp\left\{\alpha_p \int_0^t [\Phi_p[\tau] - 1] d\tau\right\}. \quad (3.5)$$

Since we have already found Φ, and hence Φ_p, this completes the exact solution of the ionization cascade problem.

4. Distribution of High-energy Electrons

A problem of interest is the calculation of the number distribution of high energy cascade electrons; that is electrons of energy $\geqq W$, where $W \gg I$ (cf. Gardner [8]). Let $q_n(W, E, t)$, $p_n(W, t)$ be the probabilities of finding after time t n electrons of energies $\geqq W$ in cascades initiated respectively by a primary knock-on electron of energy E and by the primary ionizing particle; let

$$G(z, W, E, t) = \sum_{n=0}^\infty z^n q_n(W, E, t), \quad (4.1)$$

$$H(z, W, t) = \sum_{n=0}^\infty z^n p_n(W, t), \quad (4.2)$$

be the corresponding probability generating functions. Then the expressions for G and H are obtained by making the change of variable (2.4) in the expressions for Φ,Ψ. Thus noting that $G(z,W,E,t)=1$ if $E<W$, we see from (2.22) that G can be evaluated by iteration from the relation

$$G(z,W,E,t)=\int_0^W \chi(V,E,t)dV+z\int_W^E \chi(V,E,t)dV \\ +\int_0^t d\tau\int_I^E dV\int_I^V dU G(z,W,V-U,t-\tau)G(z,W,U-I,t-\tau) \\ \varphi_i(U,V)\alpha_i(V)\chi(V,E,\tau), \tag{4.3}$$

starting the iteration in the range $W\leqq E<W+I$. However, the degradation of energy by non-ionizing collisions is negligible for electrons of energy $>W$ if $W\gg I$; hence it is legitimate in this case to use the approximation (2.15), which yields, writing $G(E,t)$ for $G(z,W,E,t)$ and remembering that $G(E,t)=1$ for $E<W$,

$$G(E,t)-1=(z-1)e^{-\alpha_i(E)t} \\ +\int_0^t e^{-\alpha_i(E)t}\alpha_i(E)d\tau\Big\{\int_I^{E-W}[G(E-U,t-\tau)-1]\varphi_i(U,E)dU \\ +\int_{W+I}^{E}[G(U-I,t-\tau)-1]\varphi_i(U,E)dU \\ +\int_{W+I}^{E-W}[G(E-U,t-\tau)-1][G(U-I,t-\tau)-1]\varphi_i(U,E)dU\Big\}, \tag{4.4}$$

with the convention that we set each of the integrals in the right-hand-sice equal to 0 if the upper limit of integration is less than the lower. Since $I\ll W$, it will be sufficient to evaluate G numerically for steps $W=jI$, $E=nI$, where j and n are large integers, $n>j$. Changing the energy variable to $\varepsilon=E/I$, we write

$$\alpha_i(n)=\alpha_n;\quad \varphi_i(k,n)=\varphi_k^n;\quad G(z,j,n,t)=g_n^j(t). \tag{4.5}$$

We then have from (4.4)

$$g_n^j(t)-1=(z-1)e^{-\alpha_n t}+\int_0^t e^{-\alpha_n t}\alpha_n d\tau \\ \Big\{\sum_{k=1}^{n-j}[g_{n-k}^j(t-\tau)-1][\varphi_k^n+\varphi_{n-k+1}^n] \\ +\sum_{k=j+1}^{n-j}[g_{n-k}^j(t-\tau)-1][g_{k-1}^j(t-\tau)-1]\varphi_k^n\Big\} \tag{4.6}$$

The first two steps in the iteration give

$$g^j_{j+1}(t) - 1 = (z-1)e^{-\alpha_{j+1}t} \tag{4.7}$$

$$g^j_{j+2}(t)-1=(z-1)\left\{e^{-\alpha_{j+2}t}+\frac{\alpha_{j+2}}{\alpha_{j+1}-\alpha_{j+2}}(e^{-\alpha_{j+2}t}-e^{-\alpha_{j+1}t})(\varphi_1^{j+2}+\varphi_{j+2}^{j+2})\right\}. \tag{4.8}$$

Finally, substituting in (3.4) and (3.5), we find that

$$H(z, W, t) = \exp.\left\{\alpha_p \int_0^t [G_p(z, W, t) - 1]d\tau\right\}, \tag{4.8}$$

where

$$G_p(z, W, t) = \int_I^\infty G(z, W, U - I, t)\varphi_p(U)dU. \tag{4.9}$$

The moments $\mu_n(W, E, t)$ of the electron initiated cascade may be obtained from G by the relation (cf. (2.6)),

$$\mu_n(W, E, t) = \left[\left(-\frac{\partial}{\partial s}\right)^n G(e^{-s}, W, E, t)\right]_{s=0}. \tag{4.9}$$

Note that $\mu_n = 0$ when $E < W$. This can be used to effect their direct numerical evaluation. Thus we find for the mean μ_1 from (4.3) and (4.4)

$$\begin{aligned}\mu_1(W, E, t) &= \int_W^E \chi(V, E, t)dV + \int_0^t d\tau \int_I^E dV \int_I^V dU[\mu_1(W, V - U, t - \tau) \\ &\quad + \mu_1(W, U - I, t - \tau)]\varphi_i(U, V)\alpha_i(V)\chi(V, E, \tau) \\ &\approx e^{-\alpha_i(E)t} + \int_0^t e^{-\alpha_i(E)\tau}\alpha_i(E)d\tau \int_{W+I}^E dU \\ &\quad [\mu_1(W, E - U, t - \tau) + \mu_1(W, U - I, t - \tau)]\varphi_i(U, E),\end{aligned} \tag{4.10}$$

The last approximation in (4.10) is, of course, valid for $W \gg I$, in which case we can also use the step evaluation introduced in (4.6), using the same notation, we have

$$\mu_1(j, n, t)=e^{-\alpha_n t}+\int_0^t e^{-\alpha_n \tau}\alpha_n d\tau \sum_{k=1}^{n-j}\mu_1(j, n-k, t-\tau)[\varphi_k^n+\varphi_{n-k+1}^n]. \tag{4.11}$$

The *cumulants* $\varkappa_n(W, t)$ of the cascade initiated by the primary ionizing particle are obtained from H by the relation

$$\begin{aligned}\varkappa_n(W, t) &= \left[\left(-\frac{\partial}{\partial s}\right)^n \log H(e^{-s}, W, t)\right]_{s=0} \\ &= \alpha_p \int_0^t d\tau \int_{W+I}^E \mu_n(W, U - I, \tau)\varphi_p(U)dU.\end{aligned} \tag{4.12}$$

The first two cumulants $\varkappa_1$, $\varkappa_2$ in particular are equal respectively to the mean and mean-square deviation of this distribution.

Since in the present case only high energy transfers invervene in the collisions with absorber atoms of both knock-on electrons and the primary particle, it is legitimate to treat the orbital atomic electrons as free: i.e., to use the 'classical' cross-section (2.8) for both α_i, φ_i and α_p, φ_p in the above calculations.

5. The Ion-Pair Distribution

The distribution of the total number of ion-pairs produced by a fast primary ionizing particle could be obtained in principle simply by setting $W = 0$ in the probability generating functions found in § 4 for the high-energy electrons. However, the approximations to be used in the two cases are different. In the ion-pair case, it is the low-energy electrons which yield the major contribution to the distribution, and hence it is important in this case to allow for energy degradation by non-ionizing collisions. Furthermore, it is a good approximation (cf. I) to assume that *each electron-initiated cascade has had the time to terminate* in a time interval t of interest; i.e. that all its end products have energies $< I$, and cannot therefore ionize further; hence the number distribution $q_n(E)$ and the corresponding probability generating function

$$G(z, E) = \sum_{n=1}^{\infty} z^n q_n(E), \tag{5.1}$$

E being as before the energy of the primary knock-on, becomes independent of t. Consequently, this G must satisfy the equation

$$\begin{aligned} &\alpha_e(E)\left\{-G(z,E)+\int_0^I G(z,E-U)\varphi_e(U,E)dU\right\} \\ &+\alpha_i(E)\left\{-G(z,E)+\int_I^E G(z,E-U)G(z,U-I)\varphi_i(U,E)dU\right\}=0, \end{aligned} \tag{5.2}$$

obtained by substituting G for Φ in (2.12) and setting $\partial G/\partial t = 0$. Its iterative solution may be found by substituting G for Φ in (2.22) and letting $t \to \infty$; thus

$$\begin{aligned} G(z,E) = z\chi_\infty(E) + \int_I^E dV \int_I^V dU G(z, V-U)G(z,U-I) \\ \varphi_i(U,V)\alpha_i(V)\bar{\chi}(V,E) \end{aligned} \tag{5.3}$$

where

$$\chi_\infty(E) = \lim_{t\to\infty} \int_0^E \chi(U, E, t)dU \tag{5.4}$$

is the probability that the primary electron will suffer no ionizing collision at any time, and

$$\bar{\chi}(V, E) = \int_0^\infty \chi(V, E, t)dt, \quad (V \geqq \mathrm{I}). \tag{5.5}$$

We start the iteration with

$$G(z, E) = z, \quad (E < I), \tag{5.6}$$

since there is exactly one ion-pair for all t when $E < I$. Let $p_n(t)$ be the number distribution of ion-pairs created in t by a fast primary particle. Since G is now independent of t, the corresponding probability generating function is from (3.4) and (3.5)

$$\begin{aligned} H(z, t) = \sum_{n=0}^{\infty} z^n p_n(t) &= \exp\{\alpha_p t[G_p(z) - 1]\} \\ &= \exp\Big\{\alpha_p t \sum_{n-1}^{\infty} (z^n - 1)q_n\Big\}, \end{aligned} \tag{5.7}$$

where

$$G_p(z) = \sum_{n=1}^{\infty} z^n q_n = \int_I^\infty G(z, U - I)\varphi_p(U)dU \tag{5.8}$$

is the probability generating function of the number distribution

$$q_n = \int_I^\infty q_n(U - I)\varphi_p(U)dU \tag{5.9}$$

of ion-pairs produced by the primary particle at each ionizing collision.

If we neglect the fluctuations of energy loss due to the non-ionizing collisions, then equation (5.2) becomes

$$\left\{a(E)\frac{\partial}{\partial E} + \alpha_i(E)\right\}G(z, E) = \alpha_i(E)\int_I^E G(z, E - U)\, G(z, U - I)\varphi_i(U, E)dU. \tag{5.10}$$

This may be obtained by substituting G for Φ in (2.29), setting $b = 0$ and $\partial G/\partial t = 0$. The solution of (5.10) is

$$G(E,z) = z\,\exp\left\{-\int_I^E \frac{\alpha_i(U)dU}{a(U)}\right\} + \int_I^E \exp\left\{-\int_W^E \frac{\alpha_i(U)dU}{a(U)}\right\} \frac{\alpha_i(W)}{a(W)}\,dW \int_I^W G(z, W-V)G(z, V-I)\varphi_i(V, W)dV. \tag{5.11}$$

The first iteration gives

$$G(E,z) = z(1-z)\,\exp\left\{-\int_I^E \frac{\alpha_i(U)dU}{a(U)}\right\} + z^2 \quad (I \leqq E < zI). \tag{5.12}$$

An improved approximation which takes into account the non-ionizing energy loss fluctuations is obtained by substituting the approximate expression (2.35) for χ in (5.3). In this case

$$\int_0^E \chi(W, E, t)dW = e^{-\lambda(t,\,E)} \tag{5.13}$$

where $\lambda(t,E)$ is defined by (2.37) and hence

$$\chi_\infty(E) = \lim_{t\to\infty} e^{-\lambda(t,\,E)}, \tag{5.14}$$

while

$$\bar{\chi}(V, E) = \int_0^\infty [2\pi\sigma^2(t, E)]^{-\frac{1}{2}}\,\exp\left\{-\frac{[E-V-m(t, E)]^2}{2\sigma^2(t, E)} - \lambda(t, E)\right\} dt \tag{5.15}$$

must be evaluated numerically.

In the present case, it is the low energy electrons which yield the major contribution to the distribution; hence small energy transfers are important. It becomes essential to allow for the binding of the orbital electrons to the nuclei in the absorber atoms, and it is therefore necessary to use the more exact quantal ionization cross-section rather than the 'classical' one.

6. Conclusions

We have obtained a general solution of the distribution problem in the theory of the ionization cascade, and we have seen that with the aid of suitable approximations, this solution can be expressed in a form that is readily amenable to numerical computation in the two cases of (a) the numbers of fast knock-on electrons, (b) the total number of ion-pairs. It is hoped that these computations will be programmed on the electronic computer that is being built in

the School of Physics of the University of Sydney, using ionization cross-sections suitable for various absorbers, and will eventually permit comparison of the theory with the available experimental material on the fluctuations in ion-pair numbers, the primary energy loss per ion-pair, and so on (cf. I). It is also hoped that the present theory will permit the evaluation of the effects of absorption at the walls or escape of knock-on electrons on the averages and fluctuations of ionization in chambers and counters [2]).

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COSMIC RAY PRODUCED BERYLLIUM ISOTOPES IN RAIN WATER

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Abstract: The two known radioactive isotopes of Beryllium (Be^7 and Be^{10}) are produced in the atmosphere in nuclear disintegrations initiated by cosmic radiation. The long lived Be^{10} (half life 2.7 million years) is difficult to detect and has not yet been isolated. The short lived isotope Be^7 (half life 53 days) is easily detected in rain water and should, therefore, be of interest in meteorological studies. The average concentration of Be^7 in 8 monsoon rains in Bombay amounted to 2400 atoms per cm^3 and is consistent with the production rates predicted by Peters[1]).

In a recent paper [1]) Peters estimated the number of radioactive nuclei of Be^{10} which must be produced in the upper atmosphere by the partial destruction of nitrogen and oxygen in collision with cosmic ray particles. It was suggested that most of the Be^{10} nuclei will reach the earth as a constituent of rain water or snow, and that the resulting activity of this isotope (a β-emission with half life of 2.7 million years) should be measurable. It could, therefore, be suitable for dating geological events which occurred during the last few million years, a period for which no radioactive dating methods are available at present.

In ref. [1]), it was estimated that in an air column of cross-section 1 cm^2, extending throughout the full height of the atmosphere, approximately 10 Be^{10} nuclei per minute should be produced at high latitudes and 2 Be^{10} nuclei at equatorial latitudes. Assuming a mean precipitation of 90 cm of water per year, rains in the two regions should on the average contain approximately 6×10^4 and 1.2×10^4 Be^{10} nuclei per cm^3, respectively. The large scale motion of air masses, however, must be expected to average out these values to some extent. A convenient method for checking these predictions consists in extracting from rain water the short lived isotope Be^7 (a γ-ray emitter with half life 53 days) which should be

produced simultaneously with Be^{10} by an analogous process and which because of its short half life is readily detectable.

In this paper we discuss experiments designed to determine the concentration of cosmic ray produced Be^7 in rain water, in the equatorial region. Arnold and Al-Salih [2]) who have independently been led to similar considerations have already established the existence of Be^7 in rain and snow falls in the Chicago area. Because of the short half life of Be^7 its presence in rain water in comparable amounts over long periods proves that it is produced continuously rather than in occasional thermo-nuclear explosions.

We have analysed eight rain samples collected during this year's monsoon in Bombay. The results show that original estimates of production rates are correct, at least as to order of magnitude. The following procedure was employed:

Approximately 100 litres of rain water were collected; the pH was adjusted to 3 by adding hydrochloric acid. The water was then passed through a cation resin exchange column (details of the experimental set-up are given in the Appendix). The resin was subsequently eluted with 1 : 2 hydrochloric acid solution.

The beryllium was then extracted from the elute by the following procedure:

After adjusting the pH to 6, and adding an excess of the sodium salt of EDTA (ethylenediamine tetra-acetic acid) to complex certain ions [3]), the samples were shaken with a 0.1 M solution of TTA in benzene for 2 hours. The beryllium along with a few other metals passes into the organic phase. This layer was then treated with concentrated HCl for a few minutes to back extract ions forming easily decomposable complexes, for example, Fe, Sr.

After destroying the organic matter by treatment with perchloric and sulphuric acids, a final purification was made by adding a 6 % solution of cupferron to the hydrochloric acid solution followed by the extraction of the impurities with chloroform.

The efficiency of the TTA extraction and the cupferron purification was checked by carrying out these steps with samples to which artificially produced Be^7 in hydrochloric acid solution had been added; the overall efficiency was found to be about 95 %.

The activity of each sample was counted in a well type scintillation spectrometer; the procedure used for measuring the efficiency of the spectrometer is described in the Appendix.

For comparison we passed a sample of tap water through the

cation exchange column and determined its activity. This water comes from an open air reservoir in the neighbourhood of Bombay. A small fraction of this water derived from rains which fell within a period of 30 days before the analysis; most of the water, however, represents precipitations during the previous monsoon (1954). The activity in the tap water was found to be much weaker than that obtained from any of the rain water samples (less than 15 %).

The 8 fairly weak individual samples were combined in order to establish the nature of the activity. By means of an end window counter we established first the absence of any β-activity. Then we determined the pulse height distribution of the emitted γ-rays by means of a single channel scintillation spectrometer. The measurements with their statistical errors are shown in Figure 1. The curve in the Figure represents the normalized pulse height distribution

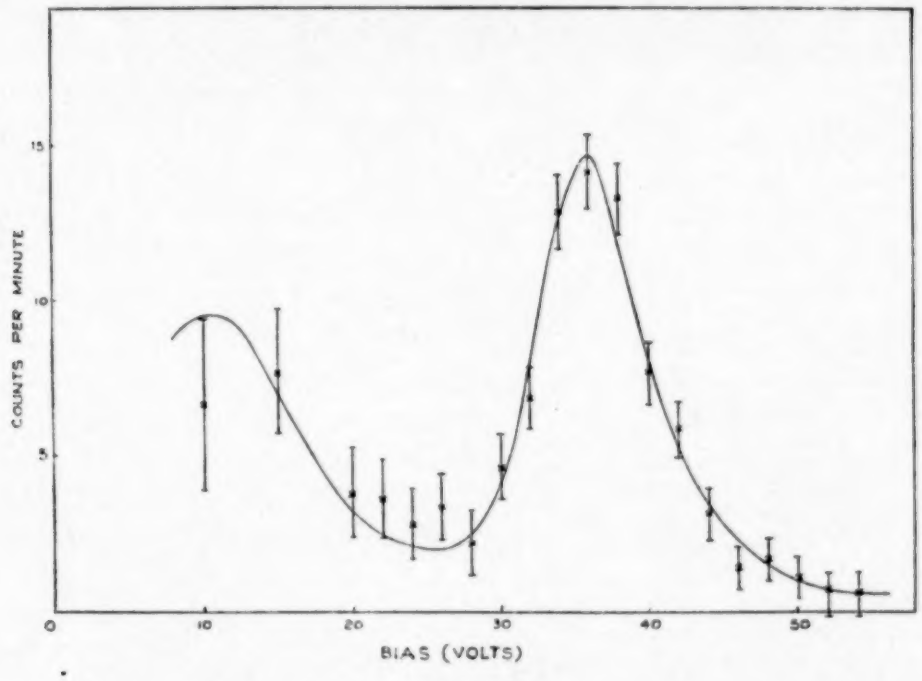

Figure 1

obtained with a strong source of artificial Be^7 under identical geometric conditions. Figure 1 shows that within the statistical error the pulses obtained from the rain water sample are identical with those obtained from Be^7 and, therefore, that its activity is consistent with the emission of a single γ-ray of 479 keV. No change in activity was observed after the sample was once again extracted with TTA and subjected to the cupferron purification process. The half life

of the activity was found to be 60 ± 10 days, and, therefore, also consistent with the conclusion that the observed activity is due to Be^7.

Using the value of half life as 53 days for the isotope Be^7 and taking account of the fact only 11 % of the disintegrations lead to the emission of 479 keV γ-ray, we deduce the following concentration of Be^7 in eight rain collections.

Mean Date of Rain Collection	Size of Sample	No. of Be^7 atoms/cm^3
5—7—55	96 litres	3.8×10^3
9—7—55	88 „	3.1 „
25—7—55	84 „	1.3 „
3—8—55	88 „	2.5 „
14—8—55	88 „	1.9 „
28—8—55	88 „	0.9 „
29—8—55	88 „	3.4 „
3—9—55	88 „	2.1 „

Our observed average Be^7 concentration is 2.4×10^3 atoms/cm^3; this should be compared with a mean value of 6×10^3 atoms/cm^3 obtained in the Chicago area by Arnold and Al-Salih [2]).

In view of the expected and observed large fluctuations in the Be^7 contents of individual rains, it is not possible as yet to state whether the lower value observed in Bombay reflects the latitude effect of the incident cosmic radiation. This could only be established by long term observations at different latitudes.

The observed concentration of Be^7 in rain water gives only a *lower limit* for the concentration of cosmic ray produced Be^{10}. The Be^{10} concentration is likely to be higher than the Be^7 concentration for two reasons:

a) Small stars (low energy nuclear disintegrations) are much more numerous in the atmosphere than large stars. The most frequent collisions are those in which only few nucleons or α-particles are ejected from the target nucleus. It is, therefore, probable that partial disintegration of nitrogen and oxygen leads more often to a residue of mass 10 than to a residue of mass 7.

b) The time necessary for the beryllium fragments to descend from the stratosphere (where most of them are formed) to the cloud forming level may be long enough to permit an appreciable fraction of Be^7 nuclei to decay in the atmosphere.

The observed average concentration of 2.4×10^3 Be^7 nuclei per cm^3 of rain water in Bombay is, therefore, perhaps not inconsistent with the expected average concentration of 1.2×10^4 Be^{10} nuclei. The production rate of Be^{10} calculated in ref. [1]) seems, therefore, to be correct at least as to order of magnitude.

It follows then that as pointed out in the earlier paper [1]) Be^{10} should be a suitable isotope for the age determination of sediments both in the oceans and on land.

It is also possible that variations in the concentration of Be^{10} in ancient ice deposits on the polar caps may give information on the climatological history of the earth.

The investigation of the Be^{10} concentration in deep ocean sediments will be reported in a separate paper.

We are grateful to Dr. J. Shankar, of the Chemistry Division, Department of Atomic Energy, for facilities provided, and to Mr. C. Lakshman Rao for his active help and co-operation during the early phases of this work. We also express our thanks to Mr. H. G. Devare and Mr. R. K. Gupta for their help with the scintillation spectrometer.

Appendix

The resin columns consisted of Dowex 50 resin (of mesh 20—50) packed in glass tubes of $1\frac{1}{2}''$ diameter. The height of the resin column was $4\frac{1}{2}''$. Two such columns were placed in series and the flow rates adjusted to 4 litres/hour. The efficiency of extraction was determined in individual runs by counting the activity of the elute in the two columns. The overall efficiency of absorption of both the columns was always found to be greater than 85 % and in 4 cases exceeded 99 %. Variations of the very large salt content in the rains collected near the sea shore are probably responsible for the variations in efficiency.

The samples were then subjected to the chemical cycle for extraction of beryllium. For counting purposes the final purified samples were evaporated to 2 cm^3 and counted on a low level scintillation spectrometer employing a NaI (TI) well type crystal of dimensions $1\frac{3}{4}''$ diameter, $2''$ height, $\frac{3}{4}''$ well diameter and $1\frac{1}{2}''$ well depth.

The overall efficiency of this crystal was calculated for a point source of 479 keV γ-rays placed 2 mm from the bottom of the well, using the Compton and the photoelectric absorption coefficients

given by Bell [5]). A similar calculation was carried out for a flat cylindrical crystal of $1\frac{1}{2}''$ diameter and $\frac{1}{2}''$ height.

The calculated detection efficiencies were 41.7 % and 8.1 % respectively.

A point source of artificial Be^7 was then constructed as follows: The sample in solution was placed on a small strip of filter paper and dried. (This paper had an ash content of $6\mu g/cm^2$). The strip of paper was wound around a thin (100μ) clean platinum wire and burnt. The wire was then heated electrically till the small ash residues melted uniformly on the wire surface. The wire was then folded to form a small source.

The activity of the wire was then counted with both types of crystal and the source strength obtained, using the calculated detection efficiencies. The two determinations disagreed by only 3.3 %.

The Be^7 was then dissolved off the platinum wire and put into 2 cm^3 of hydrochloric acid. In this form the geometry was identical with that used for the rain water samples. The activities of the samples were obtained by comparison with this standard. Counting was carried out with a 2 volt channel width at the photo peak. In this channel 3.9 % of all γ-rays are counted. Since 11 % of the disintegration of Be^7 lead to the emission of a γ-ray [6]) the number of disintegrations, D per litre, in a sample concentrated from L litres of rain water is related to the counting rate C by

$$D = \frac{C}{L \times 0.039 \times 0.11}.$$

The concentration of Be^7 atoms corresponding to the calculated disintegration rates for the 8 rain collections are given in the Table in the text.

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THEORY OF THE NEGATIVE PROTON

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Page

Historical Outline of the Investigations 202

I. Production of Negative Protons in the Pseudovector Coupling Theory

1. Production of negative protons by meson-meson collisions. 204
2. Production of negative protons by meson-nucleon collisions 209
3. Production of negative protons by proton-neutron collisions. 210
4. Loss of negative protons. 211

II. Production of Negative Protons in the Pseudoscalar Coupling Theory

5. Production of negative protons by meson-meson collisions. 213
6. Production of negative protons by meson-nuclean collisions 215
7. Production of negative protons by proton-neutron collisions. 216
8. Loss of negative protons. 220

III. The Negative Proton in Cosmic Radiation

9. Production of negative protons by nucleon-nucleus collisions 221
10. Arrival of negative protons produced by nucleon-nucleus collisions. . 222
11. The negative proton in cosmic ray showers 224
12. Production of negative protons by meson-nucleus collisions 225
13. Detectability of the negative proton in cosmic radiation 227

Abstract: A method of calculating approximately cross-sections for negative proton production according to the ps-pv and ps-ps meson theories is given. The loss of negative protons by annihilation and by meson production is discussed. An application of the results shows that it is difficult to detect the negative proton in cosmic radiation.

Historical Outline of the Investigations

The question whether there exists an elementary particle having the mass of the proton and negative electrical charge — called the negative proton or anti-proton — has claimed the attention of physicists for about twenty years. In view of the recent report that the particle has been detected it may be helpful to present the main points of the theoretical studies on the negative proton, on which the

author has been engaged at various times between 1944 and 1955.

Proton pairs may be produced from light quanta but on account of the relatively large proton mass this is an extremely rare occurrence. One therefore turned to nucleons and mesons as possible sources of negative protons. However, there were some obstacles to performing calculations which involve meson theory. The correct formulation of the theory of nuclear forces was, and is, uncertain. Besides, the perturbation method of calculating cross-sections, which gives excellent results for processes involving electrons and photons, was known to give diverging results in certain meson problems. A way towards surmounting these obstacles was shown by investigations made at the Dublin Institute for Advanced Studies between 1941 and 1943. Heitler and Peng proposed a theory of radiation damping † which gave reasonable results where the perturbation method failed. Moreover they employed the Møller-Rosenfeld theory of nuclear forces [2]) to study the production of mesons by proton-proton collisions [3]) and a subsequent application to cosmic rays [4]) seemed to explain qualitatively most of the phenomena then known.

In the early investigations on the production and annihilation of negative protons the Møller-Rosenfeld theory was adopted, but to simplify the calculations only pseudoscalar mesons were considered. The production processes examined were meson-meson and meson-nucleon collisions, and in order to avoid diverging results radiation damping methods were employed. The calculations were commenced in May 1944 and embodied in a paper [5]) read to the Royal Irish Academy in December of the same year, the paper being published the following July to be quoted as I. Two years later the problem was re-investigated by taking the complete Møller-Rosenfeld mixture of vector and pseudoscalar mesons; paper quoted as II [6]).

When these studies were made, it was generally believed that there is only one meson, which we now call the μ-meson, and that this meson is the carrier of nuclear forces. It is now known that the nuclear force mesons are π-mesons and possibly some of the heavier mesons; we consider only the π-mesons. We know that they are pseudoscalar. The nature of their coupling with the nucleon is, however, still a matter for discussion: it may be pseudovector or pseudoscalar, and we denote these possibilities by ps-pv and ps-ps, respectively. While it has been claimed that the ps-ps theory has the

† For a simplified account cf. ref. [1]).

theoretical advantage of being renormalisable, experiments on low energy meson-proton scattering performed by Fermi and his coworkers [7]) indicate that the ps-pv theory is more likely to be true. In I and II only pseudovector coupling was considered because the energy dependence of the matrix elements is then higher than in the ps-ps case, but this argument loses its weight if the ps-ps coupling constant g is much greater than the ps-pv coupling constant f or if radiation damping effects are smaller in the ps-ps than in the ps-pv theory. Accordingly both possibilities were admitted in a paper on the negative proton published in 1954 quoted as III, and the earlier calculations were extended to production by proton-neutron collisions [8]).

The discussion of the detectability of the negative proton contained in I and II is of little value, since it assumed that the nuclear force mesons are those commonly found at sea level. However, the suggestion in I that a search be made for the negative proton led Rochester and Butler to examine penetrating particles in cosmic ray showers and, while they did not discover the negative proton, they obtained one of the earliest pieces of evidence for the existence of a particle with mass about one thousand electron masses [9]). A full discussion of the probability of finding the negative proton in cosmic radiation was given in a paper read to the International Conference on Elementary Particles† held at Pisa in June 1955. It was pointed out that the most promising way of obtaining the negative proton in the atmosphere is the examination of the particles in high-energy cosmic ray showers. Nevertheless there is great difficulty in identifying a particle thus produced, so it was stated that we should rather look towards the use of high-energy accelerators to decide the question of the existence of the negative proton.

In the following sections the theory of the negative proton is developed according to the ps-pv and ps-ps theories of the π-meson.

Part I. Production of Negative Protons in the Pseudovector Coupling Theory

1. PRODUCTION OF NEGATIVE PROTONS BY MESON-MESON COLLISIONS

The simplest theoretical process in which a pair of nucleons may be produced is the collision of two oppositely charged π-mesons.

† This paper by J. Spelman and the present author, which will be quoted as IV, is to appear in a special number of *Nuovo Cimento* devoted to the Conference.

Choosing units such that $\hbar = c = \mu = 1$, where μ is the mass of the π-meson, we work in the centre-of-mass system writing $\mathbf{k}$, ε for the momentum and energy of π^+ so that $-\mathbf{k}$, ε are the momentum and energy of π^-. We denote by P, N a proton, neutron with positive energy, by P*, N* a proton, neutron with negative energy and by P^-, N^- the anti-particle formed by removing P*, N* from the negative energy state. The pair production of protons and neutrons takes place, respectively, through the transitions

$$\begin{aligned} &\pi^- + P^* \to N'; \qquad N' + \pi^+ \to P, \\ &\pi^+ + N^* \to P'; \qquad P' + \pi^- \to N. \end{aligned} \tag{1}$$

By conservation of energy and momentum, the momenta of the produced particles in the pair are equal and opposite and the total energy of each is ε. We write M for the nucleon mass, neglecting the mass-difference of the proton and neutron, and in our units $M = 6.73$. Since ε is at least Mc^2, the motion of the meson is extreme relativistic and we may replace k by ε. The matrix elements for the transitions in (1) are, respectively,

$$\sqrt{\frac{2\pi}{\varepsilon}} f(u^*_{N'} \{(\boldsymbol{\sigma}\mathbf{k}) + \varepsilon\varrho_1\} u_{P^*}); \quad \sqrt{\frac{2\pi}{\varepsilon}} f(u^*_{P} \{-(\boldsymbol{\sigma}\mathbf{k}) + \varepsilon\varrho_1\} u_{N'}),$$

where u signifies the normalised spinor of the nucleon denoted by the subscript and ϱ_1, $\boldsymbol{\sigma}$ are Dirac operators. We omit the exponential part of the wave function of the nucleon on assuming conservation of momentum for each transition, and we remark that by the orthogonality theorem

$$(u^*_P u_{P^*}) = 0.$$

The lowest order compound matrix element for the production of a proton pair from a meson pair is

$$H_{PM} = \frac{2\pi f^2}{\varepsilon} \Sigma \frac{(u^*_P \{-(\boldsymbol{\sigma}\mathbf{k}) + \varepsilon\varrho_1\} u_{N'}) (u^*_{N'} \{(\boldsymbol{\sigma}\mathbf{k}) + \varepsilon\varrho_1\} u_{P^*})}{-E_{N'}},$$

where

$$E_{N'} = \pm \sqrt{|\mathbf{p} - \mathbf{k}|^2 + M^2},$$

the energy of the neutron in the intermediate state, and $\mathbf{p}$ is the momentum of the positive proton. Σ denotes the summation over the two energy states and the two spin orientations of the neutron. This summation may be performed by the method employed in deriving the Klein-Nishina formula, viz. by using the wave equation

$$\{(\boldsymbol{\alpha} \,.\, \mathbf{p} - \mathbf{k}) + \beta M\} u_{N'} = E_{N'} u_{N'}.$$

To simplify the calculation it is necessary to distinguish between the non-relativistic (N.R.) and extreme relativistic (E.R.) cases of the proton according as $p \ll M$ or $p \gg M$; this is roughly equivalent to saying that ε is equal to M or $\varepsilon \gg M$. A short calculation then gives

$$H_{\mathrm{PM}} = \frac{4\pi f^2}{M}(u_{\mathrm{P}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{P}^*}), \qquad \text{N.R.}$$

$$H_{\mathrm{PM}} = -\frac{2\pi f^2}{\varepsilon}(u_{\mathrm{P}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{P}^*}). \qquad \text{E.R.}$$

The matrix elements for the pair production of neutrons are

$$H_{\mathrm{NM}} = -\frac{4\pi f^2}{M}(u_{\mathrm{N}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{N}^*}), \qquad \text{N.R.}$$

$$H_{\mathrm{NM}} = \frac{2\pi f^2}{\varepsilon}(u_{\mathrm{N}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{N}^*}). \qquad \text{E.R.}$$

If we consider the process

$$\pi^0 + \pi'^0 \rightarrow \mathrm{P} + \mathrm{P}^-$$

for the production of a proton pair from neutral π-mesons, we see that it occurs by both sets of transitions

$$\pi^0 + \mathrm{P}^* \rightarrow \mathrm{P}'; \qquad \mathrm{P}' + \pi'^0 \rightarrow \mathrm{P}, \qquad (2)$$

$$\pi'^0 + \mathrm{P}^* \rightarrow \mathrm{P}''; \qquad \mathrm{P}'' + \pi^0 \rightarrow \mathrm{P}. \qquad (3)$$

Since (2) and (3) give equal and opposite contributions, the compound matrix element for pair production now vanishes.

According to perturbation theory the differential cross-section for the production of proton pairs is

$$d\varphi = \frac{\pi}{v}\varrho_F \mathrm{S}\,|H_{\mathrm{PM}}|^2, \qquad (4)$$

where v is the velocity of each incoming particle, i.e. unity, S is a summation over the spin states of both P and P* and ϱ_F is the density of states in the final state. Since we have two particles in the final state,

$$\varrho_F = \frac{Mp}{16\pi^3}d\Omega, \qquad \text{N.R.}$$

$$\varrho_F = \frac{\varepsilon^2}{16\pi^3}d\Omega. \qquad \text{E.R.} \qquad (5)$$

$$\mathrm{S}\left(u_{\mathrm{p}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{p}\bullet}\right)\left(u_{\mathrm{p}\bullet}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{p}}\right)$$

$$=\frac{1}{4\varepsilon^{2}}\mathrm{tr}\,(\boldsymbol{\alpha}\mathbf{k})\{\varepsilon-(\boldsymbol{\alpha}\mathbf{p})-\beta M\}(\boldsymbol{\alpha}\mathbf{k})\{\varepsilon+(\boldsymbol{\alpha}\mathbf{p})+\beta M\}$$

$$=\frac{2}{\varepsilon^{2}}\{\varepsilon^{4}-(\mathbf{k}\mathbf{p})^{2}\}$$

$$=\begin{cases}2M^{2} & \text{N.R.}\\ 2\sin^{2}\theta\,\varepsilon^{2}, & \text{E.R.}\end{cases}$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$. Hence from (4) and (5)

$$d\varphi = 2M f^{4} p\,d\Omega, \qquad \text{N.R.}$$

$$d\varphi = \tfrac{1}{2}\sin^{2}\theta\, f^{4}\varepsilon^{2}\,d\Omega. \qquad \text{E.R.}$$

Since $d\varphi$ diverges with increasing ε, we introduce the theory of radiation damping, replacing (4) by

$$d\varphi = \frac{\pi}{v}\varrho_{F}\,\mathrm{S}\,|U_{\mathrm{PM}}|^{2}, \tag{6}$$

where U_{PM} is the solution of a number of integral equations. It may be shown† that these reduce to the comparatively simple form

$$\begin{aligned}U_{\mathrm{PM}} &= H_{\mathrm{PM}} - \pi^{2}\sum_{\mathrm{M}'}\varrho_{\mathrm{M}'}H_{\mathrm{PM}'}\sum_{\mathrm{P}'}\varrho_{\mathrm{P}'}H_{\mathrm{M}'\mathrm{P}'}U_{\mathrm{P}'\mathrm{M}}\\ &\qquad -\pi^{2}\sum_{\mathrm{M}'}\varrho_{\mathrm{M}'}H_{\mathrm{PM}'}\sum_{\mathrm{N}'}\varrho_{\mathrm{N}'}H_{\mathrm{M}'\mathrm{N}'}U_{\mathrm{N}'\mathrm{M}},\\ U_{\mathrm{NM}} &= H_{\mathrm{NM}} - \pi^{2}\sum_{\mathrm{M}'}\varrho_{\mathrm{M}'}H_{\mathrm{NM}'}\sum_{\mathrm{P}'}\varrho_{\mathrm{P}'}H_{\mathrm{M}'\mathrm{P}'}U_{\mathrm{P}'\mathrm{M}}\\ &\qquad -\pi^{2}\sum_{\mathrm{M}'}\varrho_{\mathrm{M}'}H_{\mathrm{NM}'}\sum_{\mathrm{N}'}\varrho_{\mathrm{N}'}H_{\mathrm{M}'\mathrm{N}'}U_{\mathrm{N}'\mathrm{M}}.\end{aligned} \tag{7}$$

The notation is fairly obvious, e.g. $H_{\mathrm{M}'\mathrm{P}'}$ is the matrix element for the transition from a state with a proton having momentum $\mathbf{p}'$ and a negative proton having momentum $-\mathbf{p}'$ to a state in which there are a π'^{+} with momentum $\mathbf{k}'$ and π'^{-} with momentum $-\mathbf{k}'$. The density functions are

$$\varrho_{\mathrm{M}'} = \frac{M^{2}}{16\pi^{3}}d\Omega', \quad \varrho_{\mathrm{P}'} = \varrho_{\mathrm{N}'} = \frac{Mp}{16\pi^{3}}d\Omega', \qquad \text{N.R.}$$

$$\varrho_{\mathrm{M}'} = \varrho_{\mathrm{P}'} = \varrho_{\mathrm{N}'} = \frac{\varepsilon^{2}}{16\pi^{3}}d\Omega'. \qquad \text{E.R.}$$

On substituting into (7) the tentative solutions

$$U_{\mathrm{PM}} = a\left(u_{\mathrm{p}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{p}\bullet}\right),\quad U_{\mathrm{NM}} = b\left(u_{\mathrm{N}}^{*}(\boldsymbol{\alpha}\mathbf{k})\,u_{\mathrm{N}\bullet}\right)$$

† For details of the reasoning cf. I pp. 192 and 193, and II p. 179.

we obtain

$$U_{PM} = \frac{\frac{4\pi f^2}{M}}{1 + \frac{4}{3} M^3 f^4 p} (u_P^* (\alpha \mathbf{k}) u_{P^*}), \qquad \text{N.R.}$$

$$U_{PM} = -\frac{9\pi}{f^2 \varepsilon^5} (u_P^* (\alpha \mathbf{k}) u_{P^*}). \qquad \text{E.R.}$$

Equation (6) then gives the differential cross-sections

$$d\varphi = \frac{2M f^4 p}{(1 + \frac{4}{3} M^3 f^4 p)^2} d\Omega, \qquad \text{N.R.}$$

$$d\varphi = \frac{81 \sin^2 \theta}{8 f^4 \varepsilon^6} d\Omega. \qquad \text{E.R.}$$

On integrating over the solid angle we obtain the total cross-sections

$$\varphi = \frac{8\pi M f^4 p}{(1 + \frac{4}{3} M^3 f^4 p)^2} \qquad \text{N.R.} \quad (8)$$

and

$$\varphi = \frac{27\pi}{f^4 \varepsilon^6}, \qquad \text{E.R.} \quad (9)$$

which decreases rapidly to zero for high meson energies. The same procedure gives vanishing cross-sections for the production of proton pairs from neutral mesons.

Finally we calculate the total cross-section for production of negative protons by the collision of a charged meson with energy ε' and an oppositely charged meson at rest. We bring one of the mesons considered above to rest by moving with velocity k/ε, so that the energy of the other is

$$\varepsilon' = \varepsilon \left(\varepsilon + \frac{k}{\varepsilon} k \right)$$
$$= 2\varepsilon^2 - 1,$$

or, since $\varepsilon \geqq M$,

$$\varepsilon^2 = \tfrac{1}{2}\varepsilon' \qquad (10)$$

approximately. The momentum p of the produced proton transforms according to the equation

$$p = \sqrt{\tfrac{1}{2}\varepsilon' - M^2}. \qquad (11)$$

We take $p = M$ as the dividing line for the applicability of the N.R.

and E.R. expressions for the cross-section. According to (11) this becomes $\varepsilon' = 4M^2$. In order that pair production may take place at all ε' must be at least $2M^2$. The total cross-section is a relativistic invariant and on substituting (10) and (11) into (8) and (9) we obtain expressions for the ranges indicated

$$\varphi(\varepsilon') = \frac{8\pi M f^4 \sqrt{\tfrac{1}{2}\varepsilon' - M^2}}{(1+\tfrac{4}{3}M^3 f^4 \sqrt{\tfrac{1}{2}\varepsilon' - M^2})^2}, \quad (2M^2 \leqq \varepsilon' \leqq 4M^2) \quad (12)$$

$$\varphi(\varepsilon') = \frac{216\pi}{f^4 \varepsilon'^3} \quad (\varepsilon' \geqq 4M^2) \quad (13)$$

2. PRODUCTION OF NEGATIVE PROTONS BY MESON-NUCLEON COLLISIONS

We examine the production of proton pairs by the collision of a positively or negatively charged π-meson of energy ε_0 with a neutron or proton at rest, that is,

$$\pi^+ + \mathrm{N} \to \mathrm{P} + \mathrm{P}^- + \mathrm{P}',$$

or (14)

$$\pi^- + \mathrm{P} \to \mathrm{P} + \mathrm{P}^- + \mathrm{N}'.$$

By a Lorentz transformation we bring the meson to rest, the nucleon acquiring an energy $\varepsilon_0 M$. The field of the moving nucleon is replaced by a field of virtual mesons by an adaptation of the Weizsäcker-Williams method first proposed by Heitler and Peng and later re-examined in relation to the production of π-mesons by nucleon-nucleon collisions [10]). The spectrum of the virtual mesons depends rather sensitively on $b_{\text{min.}}$, the minimum value of the impact parameter for the collision, and the study of the meson production problem indicates that we should take $b_{\text{min.}}$ equal to $\hbar/Mc$ or $\frac{1}{2}\hbar/Mc$ together with the ps-pv coupling constant f such that $f^2/\hbar c = \frac{1}{4}$. We accept this value for the coupling constant and $\hbar/Mc$ for $b_{\text{min.}}$. Then the number of virtual mesons with energy in the interval $(\varepsilon', \varepsilon' + d\varepsilon')$ is given by the approximate formula

$$q_{\text{pv}}(\varepsilon')\, d\varepsilon' = \frac{25 f^2}{\pi \varepsilon_0 M} d\varepsilon'.$$

The energy ε' is at least the rest energy 1 of the meson and its greatest value is appreciably less than the energy $\varepsilon_0 M$ of the moving nucleon. We take the greatest value to be $\frac{1}{2}\varepsilon_0 M$, though it is not possible to specify it with any accuracy. There will also be a spectrum of neutral

mesons but these will not contribute to the production of pairs.

The discussion of the processes (14) may now be reduced to that of the process examined at the end of section 1. A meson of the spectrum collides with an oppositely charged meson at rest and produces a proton pair. The total cross-section for pair production when a meson of energy ε_0 collides with a nucleon at rest is thus

$$\Phi(\varepsilon_0) = \int_{2M^2}^{\frac{1}{2}\varepsilon_0 M} \varphi(\varepsilon')\, q_{\mathrm{pv}}(\varepsilon')\, d\varepsilon',$$

where $\varphi(\varepsilon')$ is given by (12) or (13). On writing

$$\tfrac{4}{3} M^3 f^4 = a,$$

whose value is approximately 24, we find that

$$\Phi(\varepsilon_0) = \frac{800 f^6}{a^3 \varepsilon_0}\left\{ a\sqrt{\tfrac{1}{4}\varepsilon_0 M - M^2} - 2\log\left(1 + a\sqrt{\tfrac{1}{4}\varepsilon_0 M - M^2}\right) + \frac{a\sqrt{\tfrac{1}{4}\varepsilon_0 M - M^2}}{1 + a\sqrt{\tfrac{1}{4}\varepsilon_0 M - M^2}}\right\}, \quad (4M \leqq \varepsilon_0 \leqq 8M) \tag{15}$$

$$\Phi(\varepsilon_0) = \frac{625}{M^5 f^2 \varepsilon_0}\left\{1 - 0.27\left(\frac{8M}{\varepsilon_0}\right)^2\right\}. \qquad (\varepsilon_0 \geqq 8M) \tag{16}$$

The cross-sections are expressed in units $(\hbar/\mu c)^2$, i.e. 2×10^{-26} cm^2. Pair production does not occur unless the energy ε_0 is at least about 4 GeV. The cross-section rises to a maximum value of approximately 5×10^{-29} cm^2, which is reached when ε_0 is about 7 GeV, and then decreases to zero with ε_0^{-1}.

3. PRODUCTION OF NEGATIVE PROTONS BY PROTON-NEUTRON COLLISIONS

The cross-section for production of proton pairs in the process

$$\mathrm{P} + \mathrm{N} \rightarrow \mathrm{P}' + \mathrm{P}^- + \mathrm{P}'' + \mathrm{N}' \tag{17}$$

may be obtained from (15) and (16) by a further application of the Weizsäcker-Williams method. The field of the incoming proton (neutron) is resolved into a spectrum of positive (negative) π-mesons which on colliding with the neutron (proton) form a pair. Each nucleon moves with respect to the other and has a meson spectrum. However, to calculate the total cross-section due to (17) we may regard either nucleon as being at rest, the other having total energy E, say, and multiply by 2. The total cross-section for pair production by a proton-neutron collision is

$$\sigma(E) = 2\int_{4M}^{\frac{1}{2}E} \Phi(\varepsilon_0)\, q_{pv}(\varepsilon_0)\, d\varepsilon_0,$$

where

$$q_{pv}(\varepsilon_0)\, d\varepsilon_0 = \frac{25 f^2}{\pi E}\, d\varepsilon_0.$$

The threshold energy is in the neighbourhood of $8M$, that is, about 7 GeV.

On substituting the values of $\Phi(\varepsilon_0)$ and $q_{pv}(\varepsilon_0)$ we deduce that approximately

$$\sigma(E) = \frac{2.25 \times 10^4}{\pi M^6 E}\left\{2M\sqrt{\frac{E-8M}{8M}} - 2M\tan^{-1}\sqrt{\frac{E-8M}{8M}} - \frac{3}{2a}\,\frac{E-8M}{8M}\log\left(1 + aM\sqrt{\frac{E-8M}{8M}}\right)\right\}, \quad (8M \leqq E \leqq 16M)$$

$$\sigma(E) = \frac{2.25 \times 10^4}{\pi M^6 E}\left\{2M - \frac{\pi}{2}M - \frac{3}{2a}\log(1+aM)\right\} + \frac{3.1 \times 10^4}{\pi M^5 E}\left\{\log\frac{E}{16M} - 0.13\left[1 - \left(\frac{16M}{E}\right)^2\right]\right\}. \quad (E \geqq 16M)$$

The graph of $\sigma(E)$ as a function of E has a broad maximum attained when E is about 30 GeV, the maximum value being about 6×10^{-29} cm^2. This is roughly the same as the maximum value in the case of a meson-nucleon collision, and is about one thousand times smaller than the experimental cross-section $\pi(\hbar/\mu c)^2$ for meson production by a nucleon-nucleon collision when the energy of the incoming nucleon is at least 3 GeV.

4. LOSS OF NEGATIVE PROTONS

When a negative proton having energy E' and linear momentum p' collides with a proton at rest, it may be annihilated into two photons or two mesons. The cross-section for annihilation into photons is †

$$\left(\frac{e^2}{Mc^2}\right)^2 \frac{\pi}{\gamma+1}\left\{\frac{\gamma^2+4\gamma+1}{\gamma^2-1}\log(\gamma+\sqrt{\gamma^2-1}) - \frac{\gamma+3}{\sqrt{\gamma^2-1}}\right\},$$

where

$$\gamma = \frac{E'}{Mc^2}.$$

† Cf. ref. [1]) p. 270.

We may deduce that for this process the mean life-time in lead for a slow negative proton is 1.7×10^{-4} sec.

To discuss the annihilation into π-mesons we go to the centre-of-mass system, the negative proton having momentum p and energy ε and we distinguish as before between the N.R. ($p \ll M$) and E.R. ($p \gg M$) cases. By a Lorentz transformation

$$p' = \frac{\varepsilon}{M}\left(p + \frac{p}{\varepsilon}\varepsilon\right) = \frac{2\varepsilon p}{M},$$

$$E' = \frac{\varepsilon}{M}\left(\varepsilon + \frac{p}{\varepsilon}p\right) = \frac{\varepsilon^2 + p^2}{M},$$

so that

$$p' = 2p, \qquad \text{N.R.}$$

$$E' = \frac{2\varepsilon^2}{M}. \qquad \text{E.R.}$$

The annihilation of pairs might be studied as in I by solving a complicated set of integral equations for U_{MP}, which would then be inserted into the formula for the annihilation differential cross-section

$$d\varphi_{an} = \frac{\pi}{V}\varrho_F \tfrac{1}{4} S|U_{MP}|^2;$$

V is the velocity p/ε of the incoming particle †,

$$\varrho_F = \frac{\varepsilon^2}{16\pi^3}d\Omega$$

and the one quarter appears because we average over the spin states of both the positive and negative proton. On the other hand an investigation of the principle of detailed balance [11]) leads directly to the relation

$$S|U_{MP}|^2 = S|U_{PM}|^2.$$

This is sufficient for our purpose, and we shall not concern ourselves with the integral equations. We deduce immediately from (8) and (9)

$$\varphi_{an.} = \frac{2\pi M^3 f^4}{p(1 + \frac{4}{3}M^3 f^4 p)^2}, \qquad \text{N.R.}$$

$$\varphi_{an.} = \frac{27\pi}{4f^4\varepsilon^6}, \qquad \text{E.R.}$$

† V was incorrectly put equal to unity in I and this led to a wrong momentum dependence of the N.R. cross-sections.

and in the laboratory system where the positive proton is at rest

$$\varphi_{an} = \frac{4\pi M^3 f^4}{p'(1 + \frac{2}{3} M^3 f^4 p')^2}, \qquad (p' \ll 2M)$$

$$\varphi_{an} = \frac{54\pi}{M^3 f^4 E'^3}. \qquad (E' \gg 2M)$$

The cross-section diverges with p'^{-1} for small values of the momentum of the incoming negative proton and decreases steadily to zero as p' increases without limit. The mean life-time in lead of a slow negative proton is now 2×10^{-11} sec., so annihilation into mesons is about 10^7 times more likely than annihilation into photons.

When p' is about $\frac{1}{8}Mc$, the annihilation cross-section is equal to $\pi(\hbar/\mu c)^2$; that is to say, it is only for such a small value of the momentum that the annihilation cross-section becomes comparable with the experimental cross-section for meson production by a nucleon-nucleon collision. This result is important for the detectability of the negative proton. A negative proton may be lost to observation by annihilation or by meson production. If, for example it collides with a neutron, it may lose its negative charge in the transition

$$P^- + N \rightarrow \pi^- + N' + N''.$$

When the negative proton is sufficiently energetic to produce a π-meson in such a way, the cross-section for the process will be much greater than the annihilation cross-section at the same energy. It is only when the negative proton is practically brought to rest that loss by annihilation becomes important.

Part II. Production of Negative Protons in the Pseudoscalar Coupling Theory

5. PRODUCTION OF NEGATIVE PROTONS BY MESON-MESON COLLISIONS

To investigate the production of negative protons by the collision of oppositely charged π-mesons we work in the centre-of-mass system, employing our earlier notation. The matrix elements for the transitions

$$\pi^- + P^* \rightarrow N'; \qquad N' + \pi^+ \rightarrow P,$$

are, respectively,

$$-\sqrt{\frac{2\pi}{\varepsilon}}\, ig(u^*_{N'}\, \varrho_2\, u_{P*}); \quad -\sqrt{\frac{2\pi}{\varepsilon}}\, ig(u^*_{P}\, \varrho_2\, u_{N'}),$$

where ϱ_2 is the Dirac operator. We deduce that the matrix element of lowest order for pair production of protons is

$$H_{PM} = \frac{2\pi g^2}{\varepsilon}\, \frac{(u^*_P\, (\boldsymbol{\alpha}\, \mathbf{k})\, u_{P*})}{M^2 + |\mathbf{p} - \mathbf{k}|^2},$$

and this result is true for all values of p. If we consider the production of a proton pair from two neutral π-mesons, we find that the contributions to the matrix element from the transitions (2) and (3) do not cancel. However, we shall not require the matrix element for pair production by neutral mesons.

The application of perturbation methods to the matrix element H_{PM} leads to the differential cross-sections for negative proton production by two oppositely charged mesons

$$d\varphi = \frac{g^4 p}{2\varepsilon}\, \frac{M^2 + p^2 (1 - \cos^2\theta)}{\{M^2 + p^2 + k^2 - 2pk\cos\theta\}^2}\, d\Omega.$$

On integration over the solid angle we obtain the total cross-sections

$$\varphi = \frac{\pi g^4}{2M^3} p, \qquad \text{N.R.}$$

$$\varphi = \frac{\pi g^4}{2p^2}\left(\log\frac{4p^2}{M^2} - 1\right). \qquad \text{E.R.}$$

The cross-section is finite for all values of p. The total cross-section for the production of a negative proton by the collision of a meson with energy ε' and another oppositely charged meson at rest is

$$\varphi_1(\varepsilon') = \frac{\pi g^4}{2M^3}\sqrt{\tfrac{1}{2}\varepsilon' - M^2}, \qquad (2M^2 \leqq \varepsilon' \leqq 4M^2) \quad (18)$$

$$\varphi_2(\varepsilon') = \frac{\pi g^4}{\varepsilon'}\left(\log\frac{2\varepsilon'}{M^2} - 1\right). \qquad (\varepsilon' \geqq 4M^2) \quad (19)$$

We might now think of including radiation damping in our calculations. This would require the solution of integral equations somewhat more complicated than (7), and the angular dependence of the denominator of H_{PM} precludes a simple solution. From an inspection of the equations it seems that for high energies U_{PM} has the same energy dependence as H_{PM}. We therefore accept perturbation methods on the understanding that the cross-sections

may be reduced by damping. Besides, no attempt has been made to estimate the influence of radiative corrections, and these may further reduce the cross-sections.

6. PRODUCTION OF NEGATIVE PROTONS BY MESON-NUCLEON COLLISIONS

The Weizsäcker-Williams method for the ps-ps theory has been examined in a study of meson production by nucleon-nucleon collisions †. According to a charged meson theory a fast nucleon with energy E has a spectrum of virtual mesons

$$q_{\mathrm{ps}}(\bar{\varepsilon})d\bar{\varepsilon} = \begin{cases} \dfrac{A\,Mg^2\,d\bar{\varepsilon}}{\pi E} & \text{for } \bar{\varepsilon}M \leqq E \\ \dfrac{Ag^2\,d\bar{\varepsilon}}{\pi\bar{\varepsilon}} & \text{for } \bar{\varepsilon}M \geqq E, \end{cases}$$

where

$$A = 0.4, \quad g^2 = \frac{g^2}{\hbar c} \approx 4.$$

This result does not hold in a charge symmetrical meson theory, and we omit neutral mesons from our subsequent investigations.

The total cross-section for negative proton production by the collision of a meson with energy ε_0 and a nucleon at rest is

$$\Phi(\varepsilon_0) = \int_{2M^2}^{\frac{1}{2}\varepsilon_0 M} \varphi(\varepsilon')\, q_{\mathrm{ps}}(\varepsilon')\, d\varepsilon',$$

where $\varphi(\varepsilon')$ is given by (18) and (19), and

$$q_{\mathrm{ps.}}(\varepsilon')\,d\varepsilon' = \begin{cases} q'(\varepsilon')\,d\varepsilon' = \dfrac{Ag^2}{\pi\varepsilon_0}\,d\varepsilon' & \text{for } \varepsilon' \leqq \varepsilon_0 \\ q''(\varepsilon')d\varepsilon' = \dfrac{Ag^2}{\pi}\dfrac{d\varepsilon'}{\varepsilon'} & \text{for } \varepsilon' \geqq \varepsilon_0. \end{cases}$$

In calculating $\Phi(\varepsilon_0)$ we distinguish the following ranges of ε_0:

(a) $4M \leqq \varepsilon_0 \leqq 8M$, (b) $8M \leqq \varepsilon_0 \leqq 2M^2$,

(c) $2M^2 \leqq \varepsilon_0 \leqq 4M^2$, (d) $\varepsilon_0 \geqq 4M^2$,

and denote the cross-section for these ranges by $\Phi_{\mathrm{a}}(\varepsilon_0)$, $\Phi_{\mathrm{b}}(\varepsilon_0)$, $\Phi_{\mathrm{c}}(\varepsilon_0)$, $\Phi_{\mathrm{d}}(\varepsilon_0)$:

† Cf. ref. [12]) and III, pp. 54 and 55.

$$\Phi_a(\varepsilon_0) = \int_{2M^2}^{\frac{1}{2}\varepsilon_0 M} \varphi_1(\varepsilon')\, q''(\varepsilon')\, d\varepsilon'$$

$$= \frac{Ag^6}{M^2}\left\{\sqrt{\frac{\varepsilon_0 - 4M}{4M}} - \tan^{-1}\sqrt{\frac{\varepsilon_0 - 4M}{4M}}\right\},$$

$$\Phi_b(\varepsilon_0) = \int_{2M^2}^{4M^2} \varphi_1(\varepsilon')\, q''(\varepsilon')\, d\varepsilon' + \int_{4M^2}^{\frac{1}{2}\varepsilon_0 M} \varphi_2(\varepsilon')\, q''(\varepsilon')\, d\varepsilon'$$

$$= \frac{Ag^6}{M^2}\left\{1 + \tfrac{1}{4}\log 8 - \frac{\pi}{4} - \frac{2M}{\varepsilon_0}\log\frac{\varepsilon_0}{M}\right\},$$

$$\Phi_c(\varepsilon_0) = \int_{2M^2}^{\varepsilon_0} \varphi_1(\varepsilon')q'(\varepsilon')d\varepsilon' + \int_{\varepsilon_0}^{4M^2} \varphi_1(\varepsilon')q''(\varepsilon')d\varepsilon' + \int_{4M^2}^{\frac{1}{2}\varepsilon_0 M} \varphi_2(\varepsilon')q''(\varepsilon')d\varepsilon'$$

$$= \frac{\sqrt{2}g^6 A}{6M^3}\,\frac{(\varepsilon_0 - 2M^2)^{3/2}}{\varepsilon_0} + \frac{g^6 A}{M^2}\left\{1 + \tfrac{1}{4}\log 8 - \frac{\pi}{4} - \frac{2M}{\varepsilon_0}\log\frac{\varepsilon_0}{M}\right.$$

$$\left. - \sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}} + \tan^{-1}\sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}}\right\},$$

$$\Phi_d(\varepsilon_0) = \int_{2M^2}^{4M^2} \varphi_1(\varepsilon')q'(\varepsilon')d\varepsilon' + \int_{4M^2}^{\varepsilon_0} \varphi_2(\varepsilon')q'(\varepsilon')d\varepsilon' + \int_{\varepsilon_0}^{\frac{1}{2}\varepsilon_0 M} \varphi_2(\varepsilon')q''(\varepsilon')d\varepsilon'$$

$$= \frac{Ag^6}{\varepsilon_0}\left\{\tfrac{1}{2}\left(\log\frac{2\varepsilon_0}{M^2}\right)^2 - \frac{2}{M}\log\frac{\varepsilon_0}{M} + \tfrac{2}{3} + \log 8 - \tfrac{1}{2}(\log 8)^2\right\}.$$

The graph of the cross-section as a function of ε_0 rises to a broad maximum attained when ε_0 is about 30 GeV and then decreases slowly to zero. If we put $g^2/\hbar c = 4$, the maximum value is about $0.3(\hbar/\mu c)^2$, that is, somewhat less than one-tenth the cross-section for meson production by a nucleon-nucleon collision.

7. PRODUCTION OF NEGATIVE PROTONS BY PROTON-NEUTRON COLLISIONS

The total cross-section for negative proton production by the collision of a proton (neutron) with energy E and a neutron (proton) at rest is

$$\sigma(E) = 2\int_{4M}^{\frac{1}{2}E} \Phi(\varepsilon_0)\, q_{ps.}(\varepsilon_0)\, d\varepsilon_0,$$

where $\Phi(\varepsilon_0)$ denotes the cross-section derived in section 6 and

$$q_{ps}(\varepsilon_0)\,d\varepsilon_0 = \begin{cases} q_\alpha(\varepsilon_0)\,d\varepsilon_0 = \dfrac{AMg^2}{\pi E}\,d\varepsilon_0 \text{ for } \varepsilon_0 \leqq \dfrac{E}{M} \\ q_\beta(\varepsilon_0)\,d\varepsilon_0 = \dfrac{Ag^2}{\pi}\,\dfrac{d\varepsilon_0}{\varepsilon_0} \text{ for } \varepsilon_0 \geqq \dfrac{E}{M}. \end{cases}$$

The mathematical expressions are lengthy and for convenience we list the values of certain indefinite integrals, the values in some cases being approximate:

$$I_a(\varepsilon_0) = \int \Phi_a(\varepsilon_0)\,d\varepsilon_0$$
$$= \frac{Ag^6}{M^2}\left\{\frac{8M}{3}\left(\frac{\varepsilon_0-4M}{4M}\right)^{3/2} + 4M\sqrt{\frac{\varepsilon_0-4M}{4M}} - \varepsilon_0 \tan^{-1}\sqrt{\frac{\varepsilon_0-4M}{4M}}\right\},$$

$$J_a(\varepsilon_0) = \int \Phi_a(\varepsilon_0)\,\frac{d\varepsilon_0}{\varepsilon_0}$$
$$= \frac{Ag^6}{M^2}\left\{\tfrac{1}{2} - \frac{\pi}{8}\right\}\log \varepsilon_0,$$

$$I_b(\varepsilon_0) = \int \Phi_b(\varepsilon_0)\,d\varepsilon_0$$
$$= \frac{Ag^6}{M^2}\left\{\left(1 + \tfrac{1}{4}\log 8 - \frac{\pi}{4}\right)\varepsilon_0 - M\left(\log\frac{\varepsilon_0}{M}\right)^2\right\},$$

$$J_b(\varepsilon_0) = \int \Phi_b(\varepsilon_0)\,\frac{d\varepsilon_0}{\varepsilon_0}$$
$$= \frac{Ag^6}{M^2}\left\{\left(1 + \tfrac{1}{4}\log 8 - \frac{\pi}{4}\right)\log\varepsilon_0 + \frac{2M}{\varepsilon_0}\log\frac{\varepsilon_0}{M} + \frac{2M}{\varepsilon_0}\right\},$$

$$I_c(\varepsilon_0) = \int \Phi_c(\varepsilon_0)\,d\varepsilon_0$$
$$= \frac{Ag^6}{M^2}\left\{\left(1 + \tfrac{1}{4}\log 8 - \frac{\pi}{4}\right)\varepsilon_0 - M\left(\log\frac{\varepsilon_0}{M}\right)^2 - \frac{8M^2}{9}\left(\frac{\varepsilon_0 - 2M^2}{2M^2}\right)^{3/2}\right.$$
$$\left. - \frac{10M^2}{3}\sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}} + \left(\varepsilon_0 + \frac{4M^2}{3}\right)\tan^{-1}\sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}}\right\},$$

$$J_c(\varepsilon_0) = \int \Phi_c(\varepsilon_0)\,\frac{d\varepsilon_0}{\varepsilon_0}$$
$$= \frac{Ag^6}{M^2}\left\{\left(\tfrac{1}{2} + \tfrac{1}{4}\log 8 - \frac{\pi}{8}\right)\log\varepsilon_0 + \frac{2M}{\varepsilon_0}\log\frac{\varepsilon_0}{M} + \frac{2M}{\varepsilon_0}\right.$$
$$\left. + \tfrac{2}{3}\left(1 + \frac{M^2}{\varepsilon_0}\right)\sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}} - \tan^{-1}\sqrt{\frac{\varepsilon_0 - 2M^2}{2M^2}}\right\},$$

$$I_d(\varepsilon_0) = \int \Phi_d(\varepsilon_0)\, d\varepsilon_0$$
$$= \frac{Ag^6}{M^2}\left\{\frac{M^2}{6}\left(\log\frac{2\varepsilon_0}{M^2}\right)^3 - M\left(\log\frac{\varepsilon_0}{M}\right)^2 + \left[\tfrac{2}{3} + \log 8 - \tfrac{1}{2}(\log 8)^2\right] M^2 \log \varepsilon_0\right\},$$

$$J_d(\varepsilon_0) = \int \Phi_d(\varepsilon_0)\frac{d\varepsilon_0}{\varepsilon_0}$$
$$= \frac{Ag^6}{M^2}\left\{\frac{2M}{\varepsilon_0}\log\frac{\varepsilon_0}{M} - \frac{M^2}{\varepsilon_0}\log\frac{2\varepsilon_0}{M^2} - \frac{M^2}{2\varepsilon_0}\left(\log\frac{2\varepsilon_0}{M^2}\right)^2 - \frac{M^2}{\varepsilon_0}\left[\tfrac{5}{3} + \log 8 - \tfrac{1}{2}(\log 8)^2 - \frac{2}{M}\right]\right\}.$$

We must now distinguish six ranges of values of the energy E:

(i) $8M \leqq E \leqq 16M$

$$\sigma(E) = 2\int_{4M}^{\frac{1}{2}E} \Phi_a(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$= \frac{2Ag^2}{\pi}\{J_a(\tfrac{1}{2}E) - J_a(4M)\};$$

(ii) $16M \leqq E \leqq 4M^2$

$$\sigma(E) = 2\int_{4M}^{8M} \Phi_a(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0 + 2\int_{8M}^{\frac{1}{2}E} \Phi_b(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$= \frac{2Ag^2}{\pi}\{J_a(8M) - J_a(4M) + J_b(\tfrac{1}{2}E) - J_b(8M)\};$$

(iii) $4M^2 \leqq E \leqq 8M^2$

$$\sigma(E) = 2\int_{4M}^{E/M} \Phi_a(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{E/M}^{8M} \Phi_a(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0 + 2\int_{8M}^{2M^2} \Phi_b(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0 + 2\int_{2M^2}^{\frac{1}{2}E} \Phi_c(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$= \frac{2AMg^2}{\pi E}\left\{I_a\left(\frac{E}{M}\right) - I_a(4M)\right\} + \frac{2Ag^2}{\pi}\left\{J_a(8M) - J_a\left(\frac{E}{M}\right) + J_b(2M^2) - J_b(8M) + J_c(\tfrac{1}{2}E) - J_c(2M^2)\right\};$$

(iv) $8M^2 \leqq E \leqq 2M^3$

$$\sigma(E) = 2\int_{4M}^{8M} \Phi_a(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{8M}^{E/M} \Phi_b(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{E/M}^{2M^2} \Phi_b(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0 + 2\int_{2M^2}^{4M^2} \Phi_c(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{4M^2}^{\frac{1}{2}E} \Phi_d(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0.$$
$$= \frac{2AMg^2}{\pi E}\left\{ I_a(8M) - I_a(4M) + I_b\left(\frac{E}{M}\right) - I_b(8M) \right\}$$
$$+ \frac{2Ag^2}{\pi}\left\{ J_b(2M^2) - J_b\left(\frac{E}{M}\right) + J_c(4M^2) - J_c(2M^2) \right.$$
$$\left. + J_d(\tfrac{1}{2}E) - J_d(4M^2) \right\};$$

(v) $2M^3 \leqq E \leqq 4M^3$

$$\sigma(E) = 2\int_{4M}^{8M} \Phi_a(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{8M}^{2M^2} \Phi_b(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{2M^2}^{E/M} \Phi_c(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{E/M}^{4M^2} \Phi_c(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{4M^2}^{\frac{1}{2}E} \Phi_d(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$
$$= \frac{2AMg^2}{\pi E}\left\{ I_a(8M) - I_a(4M) + I_b(2M^2) - I_b(8M) \right.$$
$$\left. + I_c\left(\frac{E}{M}\right) - I_c(2M^2) \right\} + \frac{2Ag^2}{\pi}\left\{ J_c(4M^2) - J_c\left(\frac{E}{M}\right) \right.$$
$$\left. + J_d(\tfrac{1}{2}E) - J_d(4M^2) \right\};$$

(vi) $E \geqq 4M^3$

$$\sigma(E) = 2\int_{4M}^{8M} \Phi_a(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{8M}^{2M^2} \Phi_b(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{2M^2}^{4M^2} \Phi_c(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0 + 2\int_{4M^2}^{E/M} \Phi_d(\varepsilon_0)\, q_\alpha(\varepsilon_0)\, d\varepsilon_0$$
$$+ 2\int_{E/M}^{\frac{1}{2}E} \Phi_d(\varepsilon_0)\, q_\beta(\varepsilon_0)\, d\varepsilon_0$$

$$= \frac{2AMg^2}{\pi E}\left\{I_a(8M) - I_a(4M) + I_b(2M^2) - I_b(8M)\right.$$
$$\left. + I_c(4M^2) - I_c(2M^2) + I_d\left(\frac{E}{M}\right) - I_d(4M^2)\right\}$$
$$+ \frac{2Ag^2}{\pi}\left\{J_d(\tfrac{1}{2}E) - J_d\left(\frac{E}{M}\right)\right\}.$$

The asymptotic expression for the cross-section, when E is very large, is

$$\frac{A^2Mg^8}{3\pi E}\left(\log\frac{2E}{M^3}\right)^3,$$

and so the cross-section tends to zero as E increases indefinitely. The graph of the cross-section as a function of E has a broad maximum reached when E is in the neighbourhood of 200 GeV, the maximum value for $g^2/\hbar c = 4$ being about $0.45(\hbar/\mu c)^2$.

8. LOSS OF NEGATIVE PROTONS

The matrix element H_{MP} in the centre-of-mass system for the annihilation of a proton pair into a pair of oppositely charged π-mesons is the conjugate complex of H_{PM}. The annihilation cross-section calculated by perturbation theory is, in the notation of section 4,

$$\varphi_{an} = \frac{\pi g^4}{8Mp'}, \qquad \text{N.R.}$$

$$\varphi_{an} = \frac{\pi g^4}{8p^2}\left(\log\frac{4p^2}{M^2} - 1\right). \qquad \text{E.R.}$$

In the laboratory system the expressions for the annihilation cross-section are

$$\varphi_{an} = \frac{\pi g^4}{4Mp'}, \qquad (p' \ll 2M)$$

$$\varphi_{an} = \frac{\pi g^4}{4ME'}\left(\log\frac{2E'}{M} - 1\right). \qquad (E' \gg 2M)$$

The cross-section again diverges for small values of p' but the mean life-time in lead of a slow negative proton is now 2.2×10^{-9} sec. This is about one hundred times greater than that for the ps-pv case but 10^5 times smaller than the life-time for annihilation into photons. The annihilation cross-section equals $\pi(\hbar/\mu c)^2$ for

$p' = 1/11\,Mc$, and as before the loss of the negative proton by annihilation becomes important only when it is moving very slowly.

Part III. The Negative Proton in Cosmic Radiation †

9. PRODUCTION OF NEGATIVE PROTONS BY NUCLEON-NUCLEUS COLLISIONS

When studying the production of negative protons by the collision of a nucleon with an air nucleus we measure distances traversed in the atmosphere in the unit $g \cdot cm^{-2}$. The energy spectrum of the nucleons is roughly proportional to $dE/E^{2.5}$, and the number of nucleons suffers in traversing X $g \cdot cm^{-2}$ a decay represented approximately by the exponential factor $\exp(-\lambda X)$, where $\lambda^{-1} = 120$ $g \cdot cm^{-2}$. Since the cross-section for negative proton production by a proton-neutron collision is small compared with the geometrical cross-section $\pi(\hbar/\mu c)^2$ of a nucleon, the cross-section due to a nucleon-nucleus collision is the sum of the contributions from the individual nucleons in the nucleus. Thus the probability of a negative proton being produced when one nucleon with energy E passes through a distance dX is $\frac{1}{2}N\sigma(E)dX$, where $N = 6.02 \times 10^{23}$, the number of nucleons contained in 1 g, and $\sigma(E)$ is the cross-section expressed in cm^2 for negative proton production by a proton-neutron collision. The factor $\frac{1}{2}$ is present because in the theory of Parts I and II proton pairs do not result from proton-proton or neutron-neutron collisions. Let $BdE/E^{2.5}$ be the number of nucleons with energies in the range $(E, E + dE)$ which pass vertically per second across a horizontal area of 1 cm^2 at h, that is, at a depth X $g \cdot cm^{-2}$ down the atmosphere. The number of negative protons which they produce in going through 1 $g \cdot cm^{-2}$ is $\frac{1}{2}NB\sigma(E)dE/E^{2.5}$, and the total number produced per $g \cdot cm^{-2}$ is

$$n_1 = \tfrac{1}{2}NB\int_{8Mc^2}^{\infty}\frac{\sigma(E)}{E^{2.5}}\,dE. \tag{20}$$

Let us compare this with the total number of nucleons with energies not less than $3Mc^2$ which pass vertically each second across the area of 1 cm^2 at h, viz. $B\int_{3Mc^2}^{\infty}dE/E^{2.5}$. The ratio of these is

$$R = \tfrac{1}{2}N\sigma_1, \tag{21}$$

† For details of calculation and further references *vide* IV and J. Spelman, Maynooth Dissertation, 1955.

where

$$\sigma_1 = \frac{\int_{8Mc^2}^{\infty} \frac{\sigma(E)}{E^{2.5}} dE}{\int_{8Mc^2}^{\infty} \frac{dE}{E^{2.5}}};$$

R and σ_1 are independent of h.

The cross-section $\sigma(E)$ expressed in units of $(\hbar/\mu c)^2$ is given in sections 3 and 7, and on numerical integration we deduce that

$$\sigma_1 = 3.8 \times 10^{-4} \left(\frac{\hbar}{\mu c}\right)^2, \qquad \text{ps-pv (22)}$$

$$R = 2.3 \times 10^{-6}, \qquad \text{ps-pv (23)}$$

$$\sigma_1 = 5.4 \times 10^{-5} \left(\frac{g^2}{\hbar c}\right)^4 \left(\frac{\hbar}{\mu c}\right)^2, \qquad \text{ps-ps}$$

$$R = 3.3 \times 10^{-7} \left(\frac{g^2}{\hbar c}\right)^4 . \qquad \text{ps-ps}$$

We have allowed the ps-ps coupling constant g to remain because it appears in a simple way and because the numerical results depend so sensitively on its value. If we put $g^2/\hbar c = 4$,

$$\sigma_1 = 1.4 \times 10^{-2} \left(\frac{\hbar}{\mu c}\right)^2, \qquad \text{ps-ps (24)}$$

$$R = 8.5 \times 10^{-5}. \qquad \text{ps-ps (25)}$$

10. ARRIVAL OF NEGATIVE PROTONS PRODUCED BY NUCLEON-NUCLEUS COLLISIONS

The negative protons arriving at h may have been produced at any depth $X \leqq h$, and the spectrum of nucleons arriving vertically at X is

$$Be^{\lambda(h-X)} \frac{dE}{E^{2.5}}.$$

The number of negative protons produced when these nucleons traverse a thickness dX is

$$\tfrac{1}{2} NBe^{\lambda(h-X)} \frac{\sigma(E)}{E^{2.5}} dE dX.$$

Some of these are lost in the passage through the atmosphere from X to h. The loss may be due to annihilation or to meson production

and we saw in sections 4 and 8 that the loss by annihilation is relatively unimportant except for very slow negative protons. We therefore suppose that the loss is due entirely to meson production and that it is represented by the factor $\exp[-2\lambda(h-X)]$, which corresponds to a cross-section for meson production equal to the geometrical cross-section of an air nucleus. The number of negative protons which are produced in dX and reach h is thus

$$\tfrac{1}{2}NBe^{-\lambda(h-X)}\,dX\int_{8Mc^2}^{\infty}\frac{\sigma(E)}{E^{2.5}}\,dE,$$

and the total number of those which come down is

$$\tfrac{1}{2}NB\int_0^h e^{-\lambda(h-X)}\,dX\int_{8Mc^2}^{\infty}\frac{\sigma(E)}{E^{2.5}}\,dE$$

$$=\frac{NB}{2\lambda}(1-e^{-\lambda h})\int_{8Mc^2}^{\infty}\frac{\sigma(E)}{E^{2.5}}\,dE.$$

The ratio of the number of negative protons to the number of nucleons arriving at h with energies not less then 3 GeV is

$$R'=\frac{N}{2\lambda}(1-e^{-\lambda h})\sigma_1$$

$$=\frac{R}{\lambda}(1-e^{-\lambda h})$$

by (21). Hence

$$R'=\begin{cases}\dfrac{R}{\lambda}=120\,R & \text{for } \lambda h\gg 1\\ Rh & \text{for } \lambda h\ll 1.\end{cases}$$

The depth X g · cm^{-2} and the height x cm of a point are connected by the approximate relation valid for heights up to about 30,000 ft

$$X=1000e^{-bx}, \tag{26}$$

where $b=1.29\times10^{-6}$ cm^{-1}. This gives $X=1000$, $\lambda X=8.3$ at sea level and $X=650$, $\lambda X=5.4$ at the altitude of the Jungfraujoch, 11,000 ft. For a height of 100.000 ft at which balloon flights may take place it is found that $X=10$ so that $\lambda X=0.083$. Hence at sea level and at an altitude of 11,000 ft

$$R'=\begin{cases}2.8\times10^{-4} & \text{ps-pv}\\ 10^{-2} & \text{ps-ps}\end{cases}$$

and at an altitude of 100,000 ft

$$R' = \begin{cases} 2.3 \times 10^{-6} & \text{ps-pv} \\ 8.5 \times 10^{-4}. & \text{ps-ps} \end{cases}$$

11. THE NEGATIVE PROTON IN COSMIC RAY SHOWERS

Let us estimate the probability of a negative proton being present in a shower produced by a nucleon-nucleus collision. Since the threshold energy for production of proton pairs is about $8Mc^2$, we consider only those showers produced by a nucleon having at least this energy. We denote by σ_2 the cross-section $\sigma(E)$ averaged over the energy range $(8Mc^2, \infty)$ so that

$$\sigma_2 = 4.3\,\sigma_1 = \begin{cases} 1.6 \times 10^{-3}\left(\frac{\hbar}{\mu c}\right)^2 & \text{ps-pv} \\ 6 \times 10^{-2}\left(\frac{\hbar}{\mu c}\right)^2 & \text{ps-ps} \end{cases}$$

by (22) and (24). In the collisions of high energy nucleons with nuclei of atomic weight A the average cross-section for negative proton production is $\frac{1}{2}A\sigma_2$.

The cross-section σ_{sh} for the production of a shower by a nucleon-nucleus collision is equal to the geometrical cross-section of the nucleus, i.e.

$$\sigma_{sh} = \pi\left(\frac{\hbar}{\mu c}\right)^2 A^{2/3}.$$

The average number r of negative protons per shower caused by incoming nucleons with total energy above 7 GeV is therefore

$$r = \frac{\frac{1}{2}A\sigma_2}{\sigma_{sh}} = \begin{cases} \frac{8}{\pi}A^{1/3} \times 10^{-4} & \text{ps-pv} \\ \frac{3}{\pi}A^{1/3} \times 10^{-2}. & \text{ps-ps} \end{cases}$$

To form an idea of the magnitude of r we apply these results to nitrogen for a shower in the atmosphere and to silver for a shower in a photographic emulsion:

$$\text{nitrogen} \quad r = \begin{cases} 6 \times 10^{-4} & \text{ps-pv} \\ 2.3 \times 10^{-2} & \text{ps-ps} \end{cases}$$

$$\text{silver} \qquad r = \begin{cases} 1.2 \times 10^{-3} & \text{ps-pv} \\ 4.5 \times 10^{-2}. & \text{ps-ps} \end{cases}$$

12. PRODUCTION OF NEGATIVE PROTONS BY MESON-NUCLEUS COLLISIONS

The number of π-mesons with energies in the range $(\varepsilon_0, \varepsilon_0 + d\varepsilon_0)$ produced at X_0 per second in a volume determined by a horizontal area 1 cm^2 and a thickness dX_0 may be written

$$e^{-\lambda X_0} n(\varepsilon_0)\, d\varepsilon_0 dX_0, \tag{27}$$

the exponential arising from the number of incoming nucleons which in colliding with the atmospheric nuclei produce the mesons. For $\varepsilon_0 \geqq 4Mc^2$ we may express the energy spectrum as

$$n(\varepsilon_0)\, d\varepsilon_0 = C \frac{d\varepsilon_0}{\varepsilon_0^{2.5}} \tag{28}$$

and replace (27) by

$$e^{-\lambda X_0} C \frac{d\varepsilon_0}{\varepsilon_0^{2.5}} dX_0. \tag{29}$$

In traversing the atmosphere to the depth X the mesons may be lost by decay into μ-mesons and by nuclear interaction, while at the same time the energy diminishes through ionisation. Since the mesons which produce negative protons are so energetic, the influence of ionisation is small and we disregard it. Near the top of the atmosphere, e.g. at 100,000 ft altitude, the loss by decay is much more important than that by nuclear interaction. At sea level and at 11,000 ft altitude the relative importance of the two processes depends very much on the energy of the mesons. We shall neglect the loss by nuclear interaction on the understanding that at sea level and at 11,000 ft altitude our final result represents nothing better than an upper limit for the rate of negative proton production.

In terms of units of height the number of mesons decreases through the decay factor

$$e^{-\frac{\mu c(x_0 - x)}{\tau \varepsilon_0}},$$

where $\tau = 2.6 \times 10^{-8}$ sec is the life-time of the π-meson. Expressed in g · cm^{-2} by (26)

$$e^{-\frac{\mu c(x_0 - x)}{\tau \varepsilon_0}} = \left(\frac{X_0}{X}\right)^k, \tag{30}$$

where

$$k = \frac{\mu c}{\tau b \varepsilon_0} = \frac{1000\,\mu c^2}{\varepsilon_0},$$

as we see from the values of b and τ. On integration with respect to X_0 from 0 to X we deduce from (29) and (30) that the total number of mesons which arrive each second at X with energies in the interval $(\varepsilon_0, \varepsilon_0 + d\varepsilon_0)$ is

$$C \frac{d\varepsilon_0}{\varepsilon_0^{2.5}} \int_0^X e^{-\lambda X_0} \left(\frac{X_0}{X}\right)^k dX_0 = \frac{C}{\lambda} \frac{d\varepsilon_0}{\varepsilon_0^{2.5}} f(\lambda X, \varepsilon_0)$$

where

$$f(\lambda X, \varepsilon_0) = \frac{1}{(\lambda X)^k} \int_0^{\lambda X} e^{-t} t^k \, dt.$$

We next consider the production of negative protons by the collisions of these mesons with atmospheric nuclei. As in the case of nucleon-nucleus collisions, the effects of the individual nucleons in the nucleus are additive and we have a factor $\frac{1}{2}$. The number of negative protons produced per $g \cdot \mathrm{cm}^{-2}$ per sec is

$$n_2 = \frac{NC}{2\lambda} \int_{4Mc^2}^{\infty} \Phi(\varepsilon_0) f(\lambda X, \varepsilon_0) \frac{d\varepsilon_0}{\varepsilon_0^{2.5}},$$

where the cross-section $\Phi(\varepsilon_0)$ for negative proton production by a meson-nucleon collision is expressed in cm^2. Substituting the values given in sections 2 and 6 we find by numerical integration that

a) at sea level

$$n_2 = \begin{cases} 2.4 \times 10^{-9} C & \text{ps-pv} \\ 1.8 \times 10^{-8} \left(\dfrac{g^2}{\hbar c}\right)^3 C, & \text{ps-ps} \end{cases}$$

b) at 11,000 ft altitude

$$n_2 = \begin{cases} 1.3 \times 10^{-8} C & \text{ps-pv} \\ 4 \times 10^{-8} \left(\dfrac{g^2}{\hbar c}\right)^3 C, & \text{ps-ps} \end{cases}$$

c) at 100,000 ft altitude

$$n_2 = \begin{cases} 2.4 \times 10^{-8} C & \text{ps-pv} \\ 3.7 \times 10^{-8} \left(\dfrac{g^2}{\hbar c}\right)^3 C. & \text{ps-ps} \end{cases}$$

We may compare n_2 with the total number of π-mesons produced each second per $g \cdot cm^{-2}$ at X for a horizontal area of 1 cm^2. For energies below 1.5 GeV where the spectrum of (28) does not hold we use the figures of Olbert [13]) for 50° latitude and obtain the total number

$$0.14\, e^{-\lambda x}\, C.$$

We then deduce the following results:

UPPER LIMITS FOR THE RATIO OF NEGATIVE PROTON TO π-MESON PRODUCTION

	ps-pv	ps-ps
sea level	6.7×10^{-5}	$5 \times 10^{-4} \left(\frac{g^2}{\hbar c}\right)^3$
11,000 ft altitude	2×10^{-5}	$6.4 \times 10^{-5} \left(\frac{g^2}{\hbar c}\right)^3$
100,000 ft altitude	1.8×10^{-7}	$2.9 \times 10^{-7} \left(\frac{g^2}{\hbar c}\right)^3$

13. DETECTABILITY OF THE NEGATIVE PROTON IN COSMIC RADIATION

The results of the last four sections show that the probability of detecting the negative proton in cosmic radiation is very much smaller in the ps-pv than in the ps-ps theory.

We have noted that the position where the table at the end of section 12 is most likely to give an exact ratio is 100,000 ft altitude. A simple calculation shows that the rate of production of π-mesons per $g \cdot cm^{-2}$ at any depth is less than 2 per cent the total number of nucleons arriving at the same place. Combining this result with those of the table and comparing with the values of R given in (23) and (25), we deduce that near the top of the atmosphere nucleon-nucleus collisions are much more important than meson-nucleus collisions for negative proton production. Nevertheless even in the ps-ps theory the number of negative protons produced per $g \cdot cm^{-2}$ at such high altitudes is about ten thousand times smaller than the number of nucleons arriving with energies exceeding 3 GeV.

The probability of a negative proton being present in the cosmic ray showers discussed in section 11 is perhaps rather too large in our treatment of the ps-ps theory. Even in the ps-pv theory about one in every thousand high-energy showers should contain a negative proton. An examination of these showers seems to be the most

hopeful way of finding the negative proton in cosmic radiation.

The methods of Part III can be employed in a simplified form for the study of the production of negative protons by a beam of mesons or protons sent from a high-energy accelerator into a target. We then use not an averaged cross-section but $\Phi(\varepsilon_0)$ or $\sigma(E)$ evaluated for the energy of the beam. It must, of course, be remembered that the values of these given in section 2, 3, 6 and 7 will not be very accurate near the threshold energies.

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BOOK REVIEWS

H. A. BETHE and F. DE HOFFMANN, *Mesons and fields*. Vol. II (Mesons). (Row, Peterson & Co, Evanston 1955. xiii-446 pages.)

This second volume of a two-volume work on mesons and fields is the first to appear. It is mostly devoted to a thorough experimental and theoretical discussion of the properties of π-mesons; the other mesons and hyperons are only cursorily treated. After a survey of the evidence establishing the fundamental characteristics of π-mesons, an exhaustive analysis is given of the data on scattering of these mesons by nucleons and composite nuclei and on their photo-production. There follows a detailed report of various efforts to develop the theory of the coupling between nucleon and π-meson fields, having regard to the fact that the strength of this coupling is not small: this includes applications to the magnetic moments of nucleons, nuclear forces and processes of single and multiple meson production. In this connexion one regrets the absence of any reference to the ideas of Yang and Fermi, or to the intermediate coupling theory of Tomonaga.

The exposition is in the 'Handbuch', and definitely not textbook, style. Prof. Bethe's well-known virtuosity in this type of writing comes into full play, and he has no doubt been ably seconded by his distinguished collaborator. The theoretical analysis of the empirical evidence is a model of lucid and accurate presentation and acute criticism. The task the authors set themselves of making these difficult problems intelligible to experimentators untrained in the intricacies of quantum field theory has been mastered with complete success. I would even say: with too much success. Unwary readers will in fact be tempted to regard the book (at least for some time to come) as an oracle; there are even lyrical passages in it which might further encourage such an attitude. Well, they ought to know that the pronouncements they will find over merits or prospects of theories, and the way in which the emphasis is laid on the different sections strongly reflect the authors' personal views, which, authoritative as they are, cannot always claim universal assent.

Since this book is going to influence, not only research, but also the writing up of papers, in one of the most important and active domains of physics, it is a pity that it should perpetuate the illogical terminology ('electron-positron', 'isotopic spin', etc.) which disfigures many contemporary publications, and should sanction the use of Bev (instead of GeV) in violation of international convention †.

The book has been printed in England and the production is of the highest standard.

L. R.

F. MÜNZINGER, *Atomkraft* (Springer Verlag, Berlin 1955. vi-94 pages. 10.50DM.)

Der Autor, ein hervorragender Kenner des Kraftwerkbaues und des Kesselbaues, untersucht die Möglichkeiten von heute bekannten leistungsfähigen Reaktortypen, die er in einer Übersicht kurz durchspricht. Auf die Besprechung der Reaktortypen folgt eine kritische Betrachtung von Seiten der Wärmekraftmaschinen her, die für den Zusammenbau mit Reaktoren geeignet erscheinen. So gewinnt man wenigstens qualitativ eine Beurteilung ganzer Kraftwerke. Kapitel über Schutzmaßnahmen

† In an unexpected fit of erudition, the authors suggest a derivation of the word 'hyperon' from a vocable, *ὕπερον*, which has an undeniably Hellenic look, and even the right accentuation, but which is not known from Greek texts (at least not with the meaning implied): it is a produce of Ithaca, — the Ithaca of Tompkins County.

und mögliche Unfälle, sowie über die Wirtschaftlichkeit der verschiedenen Reaktoren- und Kraftwerksysteme, sind angefügt.

Das Buch ist vor der Genfer Konferenz erschienen und abgeschlossen und hat, wie der Autor auch betont, nur den Charakter einer Studie und nicht einer abschließenden Beurteilung. Für den heutigen Stand des Problems ist das Buch eine ausgezeichnete Übersicht und vor allen Dingen für den Reaktorphysiker außerordentlich lehrreich, um die Notwendigkeiten der Wärmekrafttechnik kennenzulernen.

R. Schulten

NEWS ITEMS

THE PROGRESS OF CERN

The report on the work of CERN presented by the Director General at the Fourth Session of the Council (19th-20th December) covers the period February to November 1955. It is a record of considerable achievement and progress in all the various activities. On the whole schedules both for the machines and the buildings have been well maintained.

For the Proton Synchrotron the main effort has been in developing the component parts of the vast machine and investigating the European sources of supply of the equipment. The main orders for the magnet will be placed early in 1956 and the machine is planned to come into operation in 1960. In the light of further experience a reassessment of the cost of the machine (including the buildings) has given a figure of 86 million Swiss francs.

The smaller machine, the Synchro-cyclotron is at a more advanced stage. The erection of the magnet in its building at Meyrin was started at the beginning of November. Nearly all the component parts of the Synchro-cyclotron are in the course of manufacture. Since this machine will be put into operation in 1957, the research programme for its use is now being planned.

Instruments, mainly electronic in nature and in some cases of a new design, and experimental methods to be used when the machines start working, have been developed during the year. The construction of the two Wilson cloud chambers is well advanced; the first one is actually undergoing preliminary tests after assembly.

The CERN section in Copenhagen, in addition to doing fundamental research in nuclear physics, has continued in its programme of training young theoretical physicists from Member States.

The staff of CERN, which is drawn from all the 12 participating countries has grown rapidly in pace with the expansion of the Organization. The present total is 285. In 1956 it is expected that the total staff will reach a figure of about 400.

The budgetary contributions for 1956 have been fixed at about 34 million Swiss francs as compared with a figure of 25 millions for 1955. The coming year will be the period of great constructional activity, a general move to the site at Meyrin, the installation of the laboratories and workshops, the development of the scientific programme in relation to experiments with the machines. The staff will increase by about 40 %.

By the end of 1960 CERN will probably have spent about 197 million Swiss francs — its so-called long-term budget. Previous estimates were lower but it should be realized that unforeseen problems in connection with the stability of the foundations were encountered for the Proton Synchrotron, which had to be solved by ingenious methods. Also staff needs have increased as it is now becoming clear that by 1960 the staff will probably stabilise around 500 members.

CERN SYMPOSIUM, JUNE 1956

CERN is making plans to hold a Symposium on High Energy Physics in Geneva in the period 11th — 23rd June 1956. It is hoped that the Institute of Physics of the University of Geneva will be able to place its Main Auditorium at the disposal of CERN for the Symposium.

The main subject chosen for the first week is "Novel features in the design and techniques of high energy accelerators". This will be followed by papers and discussions on special experimental techniques (e.g. bubble chambers) and new contributions to π meson physics.

Once the programme has been established invitations will be sent to specialists in this field all over the world and Member States will also be invited to send scientists. The total number of participants is to be kept at a figure not larger than 300.

The above plans were recommended by the CERN Scientific Policy Committee and finally approved by the Fourth Session of Council at its meeting in Geneva on the 19th December.

Two further recommendations were also accepted: to strengthen theoretical studies in Geneva it is proposed to invite a senior theoretical physicist for one year as a Guest Professor. He will act as Leader of the Theoretical Group in Geneva for that period. Professor G. C. Wick, Carnegie Institute of Technology, Pittsburgh has been nominated for this post.

In the same way Professor W. K. H. Panofsky, Director of the High Energy Physics Laboratory of Stanford University, has been nominated for the Guest Professorship in experimental physics for the academic year 1956—1957.

FOR PROSPECTIVE CONTRIBUTORS

Since the nomenclature in the field of Nuclear Physics is not yet quite unified and a number of illogical expressions are unfortunately still widely used, it is advisable to give the following indications about the terminology adopted in this journal.

(1) The word 'electron' applies to particles with an elementary charge of either sign. When it is desirable to emphasize the sign of the charge, the words 'positon' and 'negaton' are used. Thus the illogical phrase 'electron-positron pair' is replaced by 'electron pair', or, if required, by 'positon-negaton pair'.

(2) Nuclei (or nucleids, *not* 'nuclids') are conveniently divided according to the parity of the proton and neutron numbers Z, N, into four families as follows:

even mass nuclei	even nuclei (Z even, N even) odd nuclei (Z odd, N odd)
odd mass nuclei	odd proton nuclei (Z odd, N even) odd neutron nuclei (Z even, N odd).

(3) The expressions 'isotopic spin', 'isotopic quantum number' are replaced by 'isobaric spin', 'isobaric quantum number'.

(4) For the elementary particles use is made of the latest nomenclature adopted by international agreement.

Editors and publishers have taken steps to secure prompt publication of accepted contributions; to achieve this end they count on the co-operation of authors, who are expected, as a matter of course, to submit neatly prepared typescripts of well-written papers and to return carefully corrected proofs without delay.

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By H. C. BRINKMAN

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